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PERSONNEL RESTRAINT AND SUPPORT SYSTEM DYNAMICS

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FOREWORD

This study was initiated by the Vibration and Impact Branch, Biodynamics and Bionics Division, Biophysics Laboratory of the Aerospace Medical Research Laboratories, Wright-Patterson Air Force Base, Ohio, with the support of the Biodynamics Section, Environmental Physiology Branch, Crew Systems Division, National Aeronautics and Space Administration (NASA) - Manned Spacecraft Center, Houston, Texas. The research was conducted by Frost Engineering Development Corporation, 3910 South Kalamath Street, Englewood, Colorado 80110 under Contract AF 33(657)-9514 in support of Project 6301, "Aerospace Systems Personal Protection," Task 630102, "Personnel Protection in Aerospace Systems." Mr. Peter R. Payne was the principal investigator for Frost Engineering Development Corporation. Mr. James Brinkley of the Vibration and Impact Branch was the contract monitor for the Aerospace Medical Research Laboratories while Mr. Harris F. Scherer and Mr. Jack Rayfield were the NASA liaison representatives.

This report is one of a series of reports generated in the area of human restraint and support system dynamics under Contract No. AF 33(657)-9514. The research presented in this report was performed from July 1962 through December 1963. A general summary of this work was given by Peter R. Payne in reference 13, available from the American Society of Mechanical Engineers.

This technical report has been reviewed and is approved.

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ABSTRACT

Like any other complex dynamic system the human body responds in a complex way to acceleration inputs which vary rapidly with time. The need to avoid stresses large enough to cause injury to the body usually imposes limits on the permissible input acceleration.

The restraint system interposed between a vehicle and its occupant can modify the physiological effects of a vehicle's acceleration - time history. This modification should be made as favorable as possible by minimizing the stresses generated in the vehicle's occupant. To determine optimum dynamic characteristics for the restraint system, its important characteristics, and those of the human body, need to be represented in terms of a mathematical or "dynamic" model. Through suitable analysis, either mathematical or by means of a computer, those dynamic characteristics of the restraint system can be determined which will minimize the peak stresses developed in its human occupant.

In this report a general theory of suitable dynamic models is developed for this type of problem. Closed form solutions for a number of simple cases are presented also. In addition a method is shown which permits development of simple dynamic models for the human body utilizing existing experimental data.

Most test data has limitations. This seems particularly true when the subject is as complex and variable as the human body. The limitations associated with the application of physiological data to dynamic models of the human body can be minimized,however, if the test program is designed with this application in mind. Accordingly, the reader will find discussed in some detail, the necessary requirements for short period acceleration testing with live human subjects as well as some suggested requirements for dynamic characteristics of test rigs.

CONTENTS

LIST OF SYMBOLS

c	linear damping coefficient
$\bar{c}$	linear damping coefficient ratio, $= c/\omega$
D	the differential operator
DRI	"Dynamic Response Index" = $\omega^2 \delta_{MAX}/g$
	This is a measure of the maximum strain developed in in spring element of a dynamic model, and hence, by analogy, the human body elements which it represents.
$f(x)$	function of x
G	acceleration in g units (i e $\ddot{y}/g$)
g	acceleration due to gravity
$\bar{g}$	component of g acting parallel to axis of system
k	linear spring rate
K	linear damping force coefficient, $= cm$
m_r	value of mass r
$m_r f_s(\delta_r)$	force in spring r
$m_r f_c(\dot{\delta}_r)$	force in damper r
n	number of degrees of freedom in a dynamic system; also used as an arbitrary exponent in a nonlinear system
t	time
Δt	incremental time
Δt_1	corner duration, marking the boundary between the "impact" and "short-duration" regimes on a plot of $\ddot{y}_c$ versus Δt
Δv	velocity change

$\ddot{y}_c$	input acceleration
$\ddot{Y}_c$	peak value of $\ddot{y}_c$
y_r	vertical ordinate of mass r in a dynamic system
α, β	coefficients in the polynomial approximation of a nonlinear spring
δ	deflection of spring
δ_s	"slack distance," i e maximum separation between the occupant of a restraint system and the system itself
δ_T	total deflection of two springs in series
λ_r	unloaded length of spring r
τ	nondimensional time, $= \omega t$ for a linear system; $= \sqrt{\xi}\, t$ for a nonlinear system
ϕ_r	mass ratio, $= \frac{m_{r+1}}{m_r}$
ω	undamped natural frequency for a linear system $= \sqrt{k/m}$

Subscripts

B	condition in which bottoming occurs in a restraint system
C	input value
$CRIT$	critical value
MAX	maximum value
O	initial value
r	refers to system r, or some element thereof, in a multidegree of freedom dynamic model
T	total

Note: Dots above a symbol indicate differentiation with respect to time.

SECTION I

INTRODUCTION

This investigation is concerned generally with the stresses developed in the body of a human occupant of an aerospace vehicle which is subjected to short-period acceleration. More specifically, its objective is to ensure that the characteristics of the "restraint system," interposed between the occupant's body and the vehicle, will minimize body stresses.

To accomplish this, a means has been devised of calculating the response of the body and associated restraint system to the applied acceleration. A mathematical model has been developed which represents those characteristics of the body and restraint system which are significant in the study of the problem. Such a model necessarily involves simplification and represents only the major system considerations. Nevertheless, it has considerable value in predicting limiting stresses; indeed, the variability of human bodies and restraint system materials, may not justify the use of a more complex model.

The most elementary dynamic system in the field of restrained body dynamics, illustrated in figure 1, is mathematically quite sophisticated. Although partially closed form solutions can be obtained for a great many cases, their derivation is often so lengthy, and the resulting equations so long, that the effort involved is hardly worthwhile in an applied research program where rapid results are required.

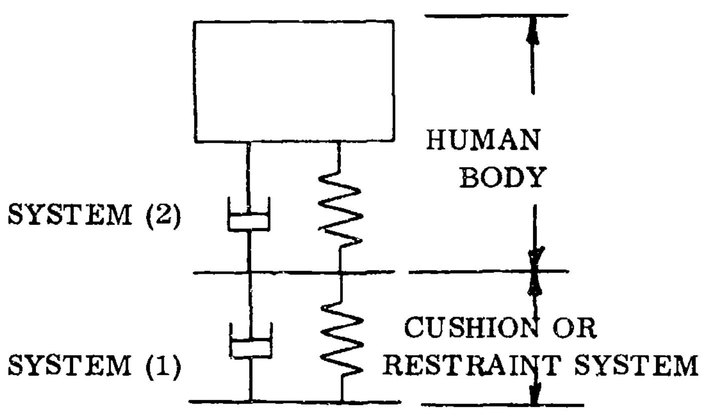


Figure 1. Elementary dynamic model in series with a restraint system.

For this reason, the influence of a particular restraint system is usually investigated with the aid of a digital or analog computer. Despite these aids of modern technology, it is still necessary to organize the basic equations of motion if we are to devise digital or analog logic of maximum economy. It is also necessary to de-

rive closed form solutions for limit cases in order that the results of computer runs can be checked during the "de-bugging" process, and to provide general design "guide-lines" as to the most promising type of system.

The basic dynamic equations involved in the problem were identified. From these equations, closed form solutions for certain limit cases were derived to permit checking of computer solutions. Concurrently, these test solutions give an insight into the basic physical behavior of the system.

No attempt is made here to explain the physical implications of the theory to the nonmathematician, since this has been done effectively elsewhere.

Simple solutions have been obtained for the influence of slack or preloading in a restraint system. These results are summarized in figures 11 and 13.

By definition, a restraint system degree of freedom consists of a mass, together with a single spring element and a single damping element. Both of these may be discontinuous and nonlinear. They are in series with a dynamic model which represents the human body. The dynamic model may have any number of degrees of freedom, both in parallel and in series. For the purposes of this report, only the series case has been considered in detail because it is the only one so far used in practice.

From the point of view of pure mathematics, there are two types of restraint systems, represented by continuous and discontinuous functions respectively. When the function is continuous and differentiable, one set of differential equations governs the entire motion. When it is discontinuous, two different sets of equations govern the motion each side of the discontinuity, and in addition, the change from one set to the other involves generating a series of initial conditions for the new equations.

In the work which follows, we shall first investigate the dynamic problem in a very general way, imposing only the requirement that the restraint functions shall be real and single-valued. Next we shall consider the special case of linear systems, and solve a number of simple cases, involving both linear restraint with discontinuities (bottoming systems) and continuous nonlinear restraint.

While it is fairly easy to determine the dynamic characteristics of a restraint system by physical measurement, the human body presents obvious difficulties. In Part II of this report we discuss this problem and show how approximate solutions can be obtained from the available test data. Test rig criteria for future experimental programs are also developed from theoretical considerations, together with the theory of bounce testing, which is suggested as a new experimental technique. It is felt that bounce testing offers hope of measuring bio-dynamic characteristics which are not easily obtained from currently employed experimental techniques.

SECTION II

DYNAMIC THEORY

1. GENERAL THEORY OF RESTRAINT DYNAMICS

a. Generalized equations for a multidegree of freedom system in series.

In body dynamics we are almost always concerned with dynamic systems in series, rather than in parallel. The springs and dampers can be either linear or nonlinear. Such a system has a number of characteristics which are independent of the characteristics of the springs and dampers. We shall deal with these before proceeding to specific relationships between force, deflection and velocity.

For the system in figure 2, let

m_r = value of the mass r

δ_r = deflection of spring r

$\dot{\delta}_r$ = velocity of spring r

$m_r f_s(\delta_r)$ = force in spring r

$m_r f_c(\dot{\delta}_r)$ = force in damper

$\phi = \frac{m_{r+1}}{m_r}$ the mass ratio.

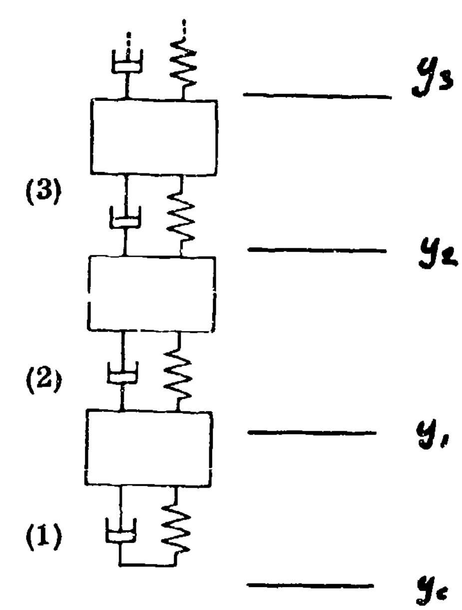


Figure 2. Series dynamic system.

The functions f_g and f_c may be any continuous, single-valued real functions, and there need be no relationship between the functions for different springs and dampers. In other words, we are dealing with a generic system.

If we now write the equations of motion for each system, the following set of coupled equations is obtained:

$$\ddot{y}_1 = f_c(\dot{\delta}_1) + f_g(\delta_1) - \phi_1 [f_c(\dot{\delta}_2) + f_g(\delta_2)]$$

$$- - - - - - - - - - - -$$

$$\ddot{y}_r = f_c(\dot{\delta}_r) + f_g(\delta_r) - \phi_r [f_c(\dot{\delta}_{r+1}) + f_g(\delta_{r+1})] \tag{1}$$

Also

$$\delta_r = \lambda_r - (y_r - y_{r-1}) \tag{2}$$

where λ_r is the unstretched spring length. Differentiating equation 2 twice with respect to time,

$$\ddot{\delta}_r = \ddot{y}_{r-1} - \ddot{y}_r \qquad \text{therefore} \qquad \ddot{y}_r = \ddot{y}_{r-1} - \ddot{\delta}_r \tag{3}$$

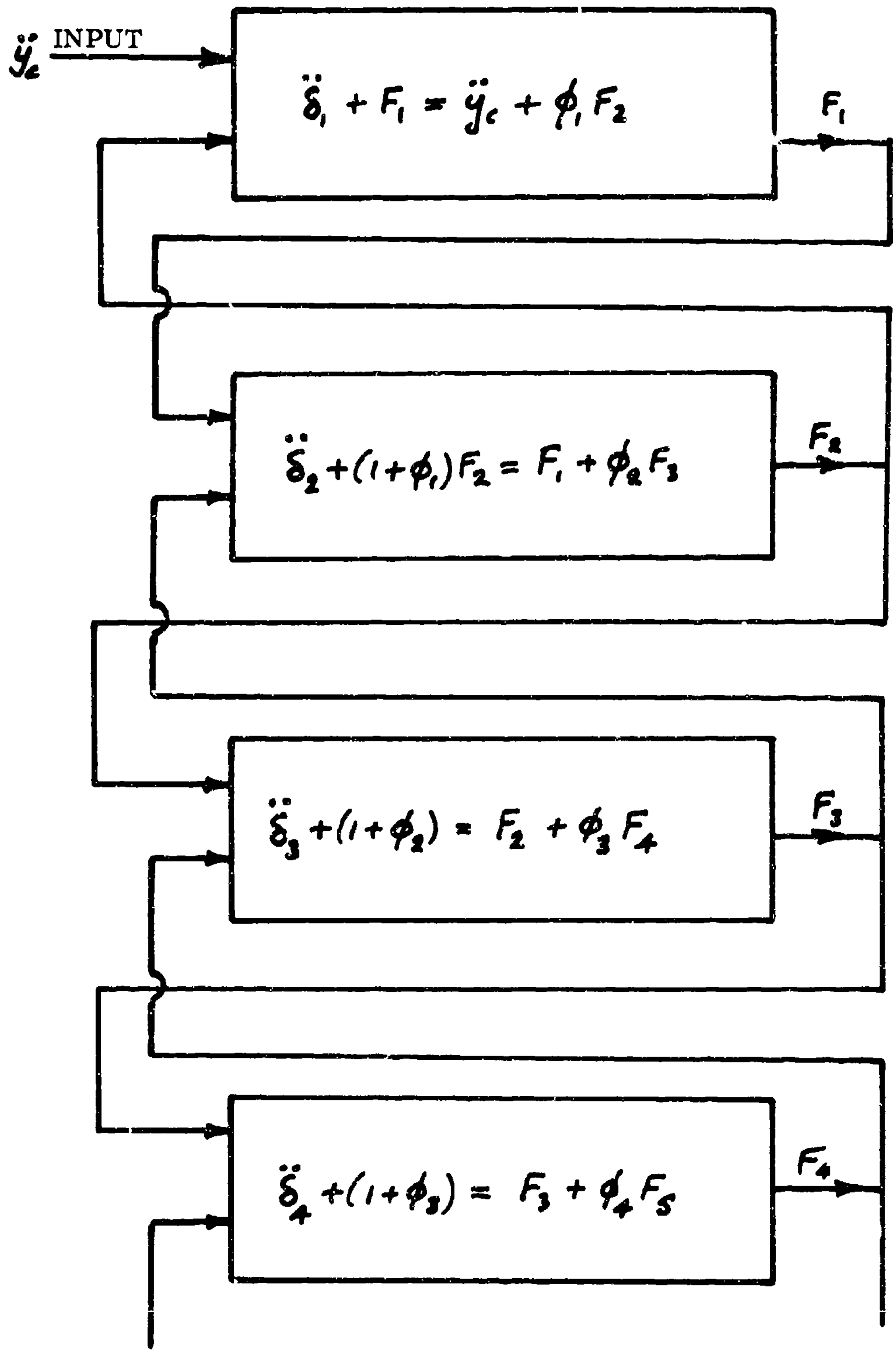
and finally

$$\ddot{y}_n = \ddot{y}_c - \sum_1^n \ddot{\delta}_r \tag{4}$$

Substituting for $\ddot{y}_r$ in equation 1,

$$\ddot{\delta}_r + [f_c(\dot{\delta}_r) + f_g(\delta_r)] = \phi_r [f_c(\dot{\delta}_{r+1}) + f_g(\delta_{r+1})] + \ddot{y}_c - \sum_1^{r-1} \ddot{\delta}_r \tag{5}$$

or $$\ddot{y}_c - \sum_1^r \ddot{\delta}_r = [f_c(\dot{\delta}_r) + f_g(\delta_r)] - \phi_r [f_c(\dot{\delta}_{r+1}) + f_g(\delta_{r+1})] \tag{6}$$



----- ETC. -----

Figure 3. Flow chart for multidegree of freedom system.

On the left of equation 5 is a conventional differential expression representing one degree of freedom. On the right, we have the force in the system immediately above it (weighted by the mass ratio ϕ_r), and an acceleration forcing term equal to the input $\ddot{y}_c$, less the accelerations of the springs between the point of application of $\ddot{y}_c$ and the system under consideration.

For convenience we can write

$$F_r = f_c(\dot{\delta}_r) + f_s(\delta_r) \tag{7}$$

so that equation 5 becomes

$$\ddot{\delta}_r + F_r = \phi_r F_{r+1} + \ddot{y}_c - \sum_{1}^{r-1} \ddot{\delta}_r \tag{8}$$

We can rearrange these equations by substituting into the right side from the previous equation, so that we express the motion of a particular system in terms only of the two systems on each side of it.

Thus,

$$\begin{aligned} \ddot{\delta}_1 + F_1 &= \ddot{y}_c + \phi_1 F_2 \\ &- - \cdot - - - - \\ \ddot{\delta}_r + (1+\phi_{r-1})F_r &= F_{r-1} + \phi_r F_{r-1} \end{aligned} \tag{9}$$

The relationships given in equation 9 are of particular importance in analog hookups, where we wish to minimize circuit complexity. The basic flow chart is shown in figure 3.

b. <u>The effect of parallel elements.</u>

If some elements occur in parallel, the generalized equations become coupled in the following manner, using the notation of figure 4.

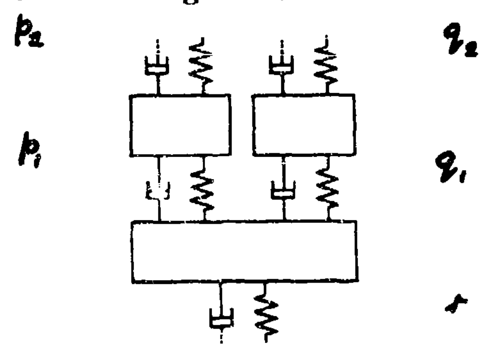


Figure 4. A branched system.

Below the branch

$$\ddot{\delta}_{r-1} + (1+\phi_{r-2})F_{r-1} = F_{r-2} + \phi_r F_r$$
$$\ddot{\delta}_r + (1+\phi_{r-1})F_r = F_{r-1} + \phi_{p_1} F_{p_2} + \phi_{q_1} F_{q_1} \tag{10}$$

Above the branch, in the p system

$$\ddot{\delta}_{p_1} + F_{p_1} = \ddot{y}_r + \phi_{p_1} F_{p_2}$$
$$\ddot{\delta}_{p_2} + (1+\phi_{p_1})F_{p_2} = F_{p_1} + \phi_{p_2} F_{p_3} \tag{11}$$

The equations are the same in the q branch, q being substituted for p .

c. General equations for a series degree of freedom without mass.

A "restraint system" can often be represented as a dynamic system in series with a dynamic model. The mass of the restraint system is considered negligible.

If the restraint system mass is finite, it can be regarded simply as system (1) in figures 2 or 3. To study this case, consider the first expression of equation 5:

$$\ddot{\delta}_1 + f_c(\dot{\delta}_1) + f_s(\delta_1) = \frac{m_2}{m_1}\left[f_c(\dot{\delta}_2) + f_s(\delta_2)\right] + \ddot{y}_c$$

by making the substitution $m_2/m_1 = \phi_1$

Then $m_1\ddot{\delta}_1 + m_1\left[f_c(\dot{\delta}_1) + f_s(\delta_1)\right] = m_2\left[f_c(\dot{\delta}_2) + f_s(\delta_2)\right] + m_1\ddot{y}_c$ (12)

Now by definition,

Force in damper $(1) = m_1 f_c(\dot{\delta}_1) = D_1$

and force in spring $(1) = m_1 f_s(\delta_1) = S_1$

(It should be noted that f_c and f_s are actually functions of $\dot{S}_1/m_1$ and S/m_1 respectively, so that D_1 and S_1 are independent of m_1.)

$$\therefore \; f_c(\dot{S}_1) = D_1/m_1$$

$$f_s(S_1) = S_1/m_1$$

Substituting into (12), we have

$$m_1\ddot{\delta}_1 + D_1 + S_1 = D_2 + S_2 + m_1\ddot{y}_c$$

Now putting $m_1 = 0$ gives $\quad D_1 + S_1 = D_2 + S_2$

i e, $\qquad F_1 = F_2$

We can approximate this condition on an analog by making ϕ_1 large compared to unity; of the order of 100 say. Then the frequency

$$\frac{k_1}{m_2} = \frac{k_1}{m_1}\left(\frac{1}{100}\right)$$

A linear restraint system can "bottom" however, and subsequent to this the acceleration input $\ddot{y}_c$ is fed directly to the dynamic model, together with an impulsive velocity change which is equal to the velocity at which the restraint spring bottomed. This latter requirement involves separation and the differentiation of velocity signals, and it is found that for analog work, more conventional methods of programming are preferred.

For nonlinear restraint systems, however, this system can be used very effectively since there is no discontinuity in the equations of motion.

When the mass of the restraint system can be neglected, the dynamic model is as shown in figure 5. Equation 8 still applies to this system, except that the terms for $r = 1$ are replaced by the simple force equality

$$F_1 = F_2 \tag{13}$$

Thus the equations become

$$\begin{aligned} F_1 &= F_2 \\ \ddot{\delta}_2 + F_2 &= \phi_2 F_3 + (\ddot{y}_c - \ddot{\delta}_1) \\ &- \; - \; - \; - \; - \; - \\ \ddot{\delta}_r + (1+\phi_{r-1})F_r &= F_{r-1} + \phi_{r+1}F_r \end{aligned} \tag{14}$$

$\ddot{\delta}_1$ is an unknown which must be determined from the first equation. Substituting equation 7 for F_1 and F_2 ,

$$f_c(\dot{\delta}_1) + f_s(\delta_1) = f_c(\dot{\delta}_2) + f_s(\delta_2) \tag{15}$$

By differentiating this equation with respect to time it is often possible to obtain a solution for δ_1 in terms of δ_2 .

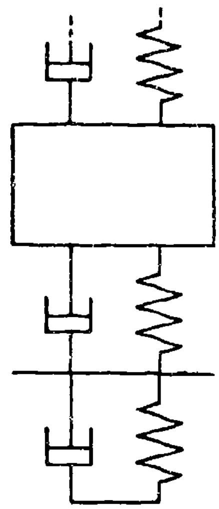


Figure 5. Zero mass restraint system in series with a dynamic model.

From the foregoing it is clear that we may express the influence of zero mass restraint system in the following theorem.

> "The only influence of a zero mass restraint system on the equations of a dynamic model is to replace the forcing term $\ddot{y}_c$ with the term ($\ddot{y}_c - \ddot{\delta}_1$), wherever it appears."

2. LINEAR SYSTEMS

For a linear system, let

$$2K_r = \frac{\text{damper force}}{(\dot{\delta}_r)}, \quad c_r = \frac{K_r}{m_r} = \frac{f_c(\dot{\delta}_r)}{(\dot{\delta}_r)}$$

$$k_r = \frac{\text{spring force}}{(\delta_r)}, \quad \omega^2 = \frac{k_r}{m_r} = \frac{f_s(\delta_r)}{(\delta_r)}$$

Then equation 5 becomes

$$\ddot{\delta}_r + 2c_r\dot{\delta}_r + \omega_r^2\delta_r = \phi_r\left\{2c_{r+1}\dot{\delta}_{r+1} + \omega_{r+1}^2\delta_{r+1}\right\} + \ddot{y}_c - \sum_1^{r-1}\ddot{\delta}_r \tag{16}$$

Equation 9 becomes

$$\ddot{\delta}_r + (1+\phi_{r-1})\left\{2c_r\dot{\delta}_r + \omega_r^2\delta_r\right\} = \left\{2c_{r-1}\dot{\delta}_{r-1} + \omega_{r-1}^2\delta_{r-1}\right\} + \phi_r\left\{2c_{r+1}\dot{\delta}_{r+1} + \omega_{r+1}^2\delta_{r+1}\right\} \tag{17}$$

When the excitation ($\ddot{y}_c$) is sinusoidal, or can be resolved into a Fourier series, equation 17 can be solved using conventional impedance techniques. Alternatively, we can use Laplace and Fourier transformations, which convert the differential equations into algebraic equations. For example the Laplace transformation of equation 16 is

$$\mathcal{L}(\delta_r)\left\{p^2 + 2c_rp + \omega_r^2\right\} = \mathcal{L}(\delta_{r+1})\phi_r\left\{2c_{r+1}p + \omega_{r+1}^2\right\} + \mathcal{L}(\ddot{y}_c) - p^2\left\{\mathcal{L}(\delta_1) + \mathcal{L}(\delta_2) + \mathcal{L}(\delta_{r-1})\right\}$$

$$+ \text{ initial conditions.} \tag{18}$$

This set of simultaneous equations can be solved for $\mathcal{L}(\delta)$. An analytical expression for $\mathcal{L}(\delta)$ can be inverted to $\delta = f(t)$ by means of tables of Laplace transforms, or the integral

$$\delta = \frac{1}{2\pi i} \oint F(p) e^{pt} \, dp$$

where $\oint$ denotes integration about a closed contour.

3. CLOSED FORM SOLUTIONS FOR SPECIAL LINEAR CASES

a. Wholly viscous restraint.

In this section we consider the idealized case of a restraint system whose mass and spring characteristics are negligible compared with its damping characteristics, as indicated in figure 6.

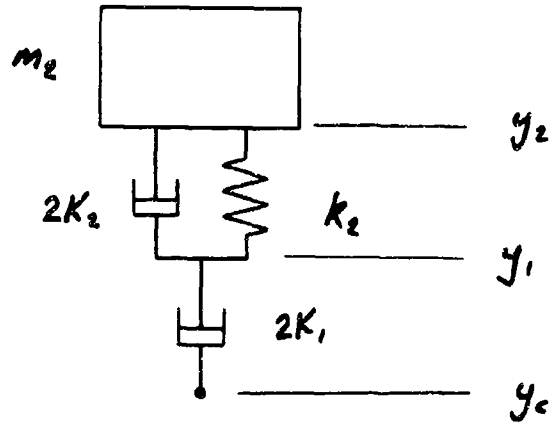


Figure 6. Wholly viscous restraint system.

Equation 16 then simplifies to

$$\left.\begin{aligned} \ddot{\delta}_2 + 2c_2\dot{\delta}_2 + \omega_2^2\delta_2 &= \ddot{y}_c - \ddot{\delta}_1 \\ 2c_2\dot{\delta}_2 + \omega_2^2\delta_2 &= 2c_1\dot{\delta}_1 \end{aligned}\right\} \quad (t < t_6) \tag{19}$$

Differentiating the restraint equation

$$\ddot{\delta}_1 = \frac{c_2}{c_1}\ddot{\delta}_2 + \frac{\omega_2^2}{2c_1}\dot{\delta}_2 \tag{20}$$

Substituting for $\ddot{\delta}_1$ in the first of equation 19, the equation of motion becomes

$$\ddot{\delta}_2 + \frac{\left[2c_2 + \frac{\omega_2^2}{2c_1}\right]}{(1 + c_2/c_1)}\dot{\delta}_2 + \frac{\omega_2^2}{(1 + c_2/c_1)}\delta_2 = \frac{\ddot{y}_c}{(1 + c_2/c_1)} \qquad (t < t_6) \tag{21}$$

We have thus reduced a two degree of freedom problem to a single degree of freedom equation in δ_2. We know that

if
$$\frac{\frac{2c_2}{\omega_2} + \frac{\omega_2}{2c_1}}{\sqrt{1 + c_2/c_1}} < 2.0 \tag{22}$$

the motion will be subcritically damped.

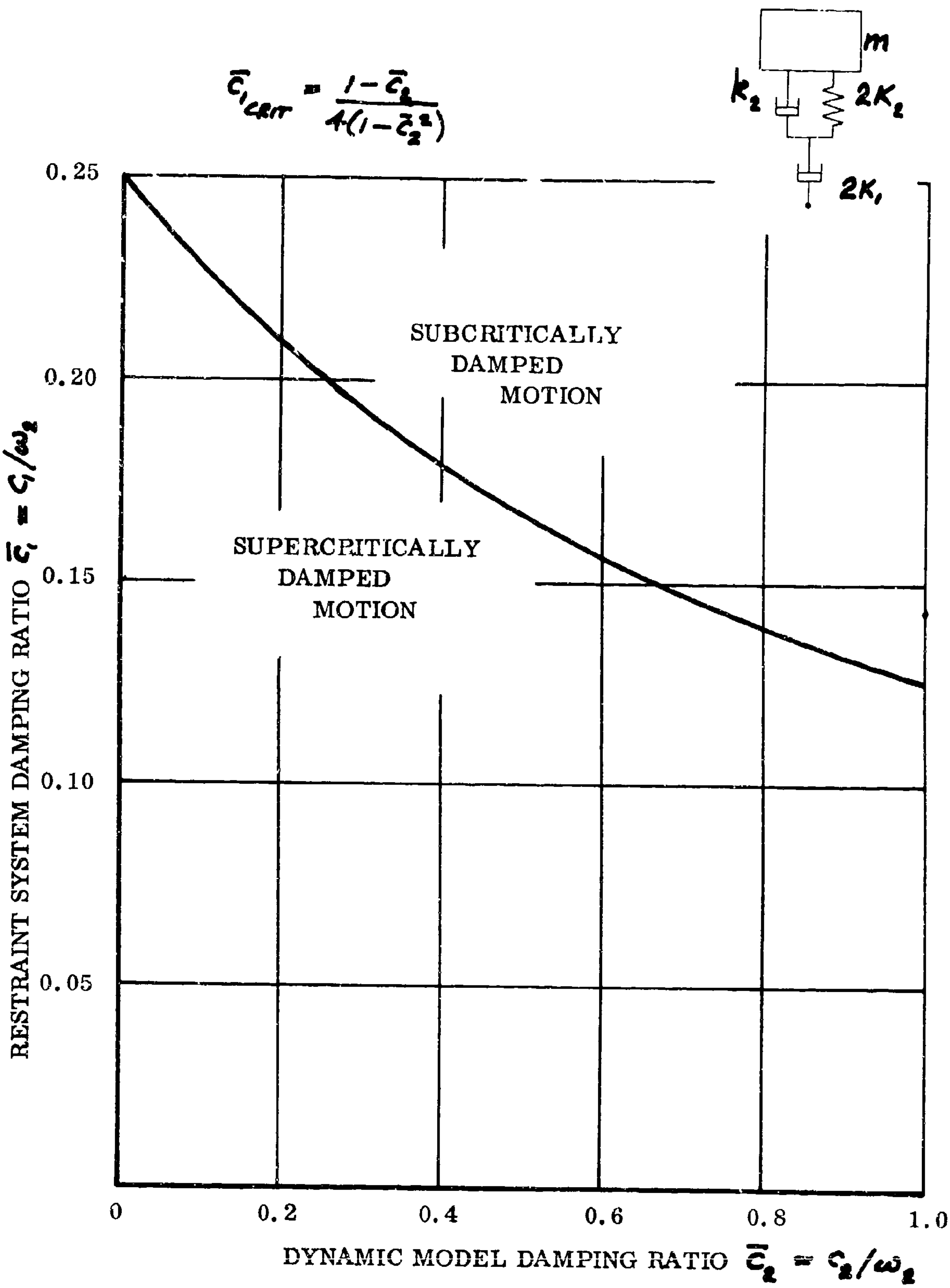


Figure 7. Restraint system damping required to give "Dead-Beat" motion of linear dynamic model.

Insofar as dynamic overshoot of the human body is a problem, at least for short period, short rise time pulses, a "deadbeat" restraint system is of considerable interest. From equation 22 this could be achieved if

$$4\bar{c}_2^2 + 2\frac{\bar{c}_2}{\bar{c}_1} + \frac{1}{4\bar{c}_1^2} = 4\left(1 + \bar{c}_2/\bar{c}_1\right)$$

where $\bar{c}_1 = c_1/\omega_2$, $\bar{c}_2 = c_2/\omega_2$

$$\therefore \quad \bar{c}_{1_{CRIT}} = \frac{1 - \bar{c}_2}{4(1 - \bar{c}_2^2)} \qquad (23)$$

Equation 23 is plotted in figure 7, and a table of values is given below.

$\bar{c}_2$	$\bar{c}_{1_{CRIT}}$	$\bar{c}_2$	$\bar{c}_{1_{CRIT}}$
.0	.25	.6	.156
.05	.238	.7	.147
.1	.228	.8	.139
.2	.208	.9	.132
.3	.192	.95	.128
.4	.179	1.0	.125
.5	.167		

Closed form solutions to the complete problem cannot be obtained because it is impossible to obtain a closed form solution to the equation for time to bottom-out the restraint system. This point is best illustrated by considering the even simpler system of figure 8, where the damping of the human body dynamic model is assumed to be zero.

This simplification permits the auxiliary (restraint) equation to be written as

$$k_2 \delta_2 = 2K_1 \dot{\delta}_1 \qquad (24)$$

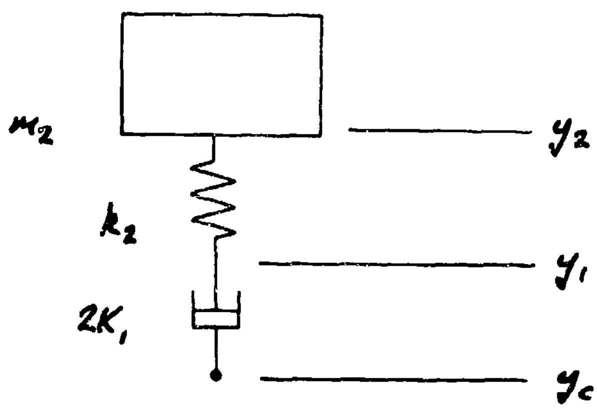


Figure 8. Dynamic system for a zero damping model.

Defining $\omega^2 = k_2/m_2$, $c = K_1/m_2$ $\bar{c} = c/\omega$

$$\omega^2 \delta_2 = 2c \frac{d\delta_1}{dt} \tag{25}$$

or

$$\delta_1 = \frac{\omega}{2\bar{c}} \int_0^t \delta_2 \, dt \tag{26}$$

The basic equations of motion of the system are then

$$\ddot{\delta}_2 + \frac{\omega}{2\bar{c}} \dot{\delta}_2 + \omega^2 \delta_2 = \ddot{y}_c \qquad (t < t_B)$$

$$\ddot{\delta}_2 + \omega^2 \delta_2 = \ddot{y}_c \qquad (t > t_B) \tag{27}$$

where t_B is the time at which the restraint system bottoms out ($\delta_1 = \delta_{1B}$).

In terms of a nondimensional time $\tau = \omega t$

$$\frac{d\delta}{dt} = \omega \frac{d\delta}{d\tau} \qquad \frac{d^2\delta}{dt^2} = \omega^2 \frac{d^2\delta}{d\tau^2}$$

The equations of motion become

$$\frac{d^2\delta_2}{d\tau^2} + \frac{1}{2\bar{c}} \frac{d\delta_2}{d\tau} + \delta_2 = \frac{d^2 y_c}{d\tau^2} \qquad (\tau < \tau_B)$$

$$\frac{d^2\delta_2}{d\tau^2} + \delta_2 = \frac{d^2 y_c}{d\tau^2} \tag{28}$$

and from equation 26

$$\delta_1 = \frac{1}{2\bar{c}} \int_0^\tau \delta_2 \, d\tau \tag{29}$$

From figure 7 we see that if

$\bar{c} < 1/4$, the damping is subcritical.
$\bar{c} > 1/4$, the damping is supercritical.

(1) <u>Response to an impulsive velocity change $\Delta\bar{v}$.</u>

When the forcing function is zero, and the initial velocity $(\dot{\delta})_0 = \Delta\bar{v}$, equation 28 has the solution

$$\frac{\delta_2}{\Delta\bar{v}} = \frac{e^{-\tau/4\bar{c}}}{\eta} \sin \eta\tau \tag{30}$$

$$\frac{\delta_{2MAX}}{\Delta\bar{v}} = e^{-\varphi/4\bar{c}\eta} \tag{31}$$

where $\varphi = \sin^{-1}\eta$ in the second quadrant.

$$\eta = \sqrt{1 - 1/16\bar{c}^2}$$

Also, from equations 29 and 30

$$\frac{\delta_1}{\Delta\bar{v}} = \frac{1}{2\bar{c}}\left[1 - \frac{e^{-\tau/4\bar{c}}}{\eta} \sin(\eta\tau + \varphi)\right] \tag{32}$$

We obtain the maximum value of δ_1 by differentiating equation 32 with respect to τ.

$$\frac{1}{\Delta\bar{v}}\left(\frac{d\delta_1}{d\tau}\right) = \frac{-e^{-\tau/4\bar{c}}}{2\bar{c}\eta} \sin(\eta\tau + \varphi + \theta) \tag{33}$$

where $\sin\theta = +\eta$

i e, $\theta = +\varphi$ or $n\pi - \varphi$

δ_1 is a maximum when $\eta\tau_M + \varphi = \pi n - \varphi$

$$\therefore \tau_M = 2\varphi/\eta$$

$$\frac{\delta_{1MAX}}{\Delta\bar{v}} = \frac{1}{2\bar{c}}\left[1 - e^{-2\varphi/4\bar{c}\eta}\right] \tag{34}$$

It is therefore possible to plot curves of $\omega \delta_{2_{MAX}} / \Delta v$ as a function of $\bar{\epsilon}$, and also curves of $\omega \delta_{1_{MAX}} / \Delta v$. From these we can determine the variation of $\delta_{2_{MAX}}$ with allowable bottoming depth $\delta_{1_{MAX}}$.

i e

$$\frac{\omega^2 \delta_{2_{MAX}}}{\omega \Delta v} = e^{-\varphi/4\bar{\epsilon}\eta} \tag{35}$$

$$\frac{\omega^2 \delta_{1_{MAX}}}{\omega \Delta v} = \frac{1}{2\bar{\epsilon}}\left[1 - e^{-2\varphi/4\bar{\epsilon}\eta}\right] \tag{36}$$

(2) Response to a zero rise time acceleration $\ddot{Y}_c$.

For this case the right side of equation 28 becomes

$$\bar{Y}_c = \ddot{Y}_c / \omega^2$$

The solutions are

$$\frac{\delta_2}{\bar{Y}_c} = 1 - \frac{e^{-\tau/4\bar{\epsilon}}}{\eta} \sin(\eta\tau + \varphi) \tag{37}$$

where $\varphi = \sin^{-1}\eta$

$$\frac{\delta_{2_{MAX}}}{\bar{Y}_c} = 1 + e^{-\pi/4\bar{\epsilon}\eta} \tag{38}$$

The restraint deflection is

$$\frac{\delta_1}{\bar{Y}_c} = \frac{1}{2\bar{\epsilon}} \int_0^\tau \left[1 - \frac{e^{-\tau/4\bar{\epsilon}}}{\eta} \sin(\eta\tau + \varphi)\right] d\tau$$

and after some manipulation this becomes

$$\frac{\delta_1}{\bar{Y}_c} = \frac{1}{2\bar{\epsilon}}\left(\tau - \frac{1}{2\bar{\epsilon}}\right) + \frac{e^{-\tau/4\bar{\epsilon}}}{\eta} \sin(\eta\tau + \varphi) \tag{39}$$

In this case there is no true maximum deflection since δ_1 continually increases, and it is therefore logical to define the critical bottoming depth as the value of δ_1 when δ_2 reaches its maximum.

That is, when

$$\frac{e^{-\tau_m/4\bar{c}}}{\eta} \sin(\eta\tau_m + \varphi + \theta) = 0$$

where $\theta = -\sin^{-1}\eta = -\varphi$

$$\therefore \frac{\tau_m}{4\bar{c}} = \frac{n\pi}{4\bar{c}\eta}. \tag{40}$$

$$\therefore \frac{\delta_{2MAX}}{\bar{Y}_c} = 1 + e^{-\pi/4\bar{c}\eta} \tag{41}$$

Substituting equation 40 for τ_m in equation 39

$$\frac{\delta_{1MAX}}{\bar{Y}_c} = \frac{1}{\bar{c}}\left(\frac{\pi}{\eta} - \frac{1}{2\bar{c}}\right) + e^{-\pi/4\bar{c}\eta} \tag{42}$$

Note than when $\bar{c} \gg 1.0$ $\quad \eta \rightarrow 1.0$

$$\frac{\delta_{2MAX}}{\bar{Y}_c} \rightarrow 2.0$$

$$\frac{\delta_{1MAX}}{\bar{Y}_c} \rightarrow \left(1 + \frac{\pi}{4\bar{c}}\right)$$

b. <u>Linear Spring Restraint.</u>

The corollary to the idealization of Section a. is the system illustrated in figure 9. For this case equation 16 becomes

$$\ddot{\delta}_2 + 2c_2\dot{\delta}_2 + \omega_2^2\delta_2 = \ddot{y}_c - \ddot{\delta}_1$$

$$2c_2\dot{\delta}_2 + \omega_2^2\delta_2 = \omega_1^2\delta_1 \tag{43}$$

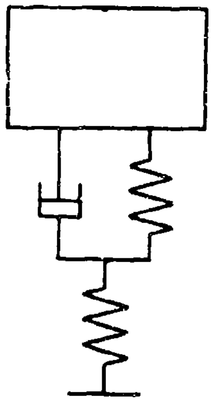


Figure 9. Linear spring restraint.

Differentiating the restraint equation

$$\ddot{\delta}_1 = \left(\frac{\omega_2}{\omega_1}\right)^2 \ddot{\delta}_2 + \frac{2c_2}{\omega_1^2} \dddot{\delta}_2 \tag{44}$$

The presence of the third derivative makes any solution difficult, so we will once again use the zero-damping model, by writing $c_2 = 0$.

Defining $\varphi = (\omega_2/\omega_1)^2$

$$\ddot{\delta}_2 + \frac{\omega_2^2}{1+\varphi}\,\delta_2 = \frac{\ddot{y}_c}{1+\varphi} \tag{45}$$

and

$$\delta_1 = \varphi\,\delta_2 \tag{46}$$

(1) <u>Solution for short period ($\Delta t > \Delta t_1$) acceleration with zero rise time.</u>

We assume that the acceleration input rises instantly from zero to $\ddot{y}_c = \ddot{Y}_c$. Then from equation I. 3 in Appendix I

$$\frac{\omega_2^2\delta_2}{1+\varphi}\cdot\frac{1+\varphi}{\ddot{Y}_c} = \frac{\omega_2^2\delta_2}{\ddot{Y}_c} = \frac{\omega_1^2\delta_1}{\ddot{Y}_c} = 1-\cos\frac{\omega_2 t}{\sqrt{1+\varphi}} \tag{47}$$

$$\therefore \cos\frac{\omega_2 t_B}{\sqrt{1+\varphi}} = 1-\frac{\omega_1^2\delta_{1B}}{\ddot{Y}_c} \tag{48}$$

let $\omega_1^2 \delta_{1B} / \ddot{Y}_c = \xi$

Now

$$\frac{\omega_2^2 \dot{\delta}_2}{\ddot{Y}_c} = \frac{\omega_2}{\sqrt{1+\varphi}} \sin \frac{\omega_2 t}{\sqrt{1+\varphi}} = \frac{\omega_2}{\sqrt{1+\varphi}} \sqrt{1 - \cos^2 \frac{\omega_2 t}{\sqrt{1+\varphi}}}$$

$$\frac{\omega_2 \dot{\delta}_{2B}}{\ddot{Y}_c} = \frac{1}{\sqrt{1+\varphi}} \sqrt{2\xi - \xi^2} \tag{49}$$

and of course, from equation 46

$$\delta_{2B} = \delta_{1B} / \varphi$$

so that

$$\frac{\omega_2 \dot{\delta}_{1B}}{\ddot{Y}_c} = \frac{\varphi}{\sqrt{1+\varphi}} \sqrt{2\xi - \xi^2} \tag{50}$$

Therefore the total (initial condition) spring velocity after bottoming will be

$$\frac{\omega_2 \dot{\delta}_{1B}}{\ddot{Y}_c} = \frac{\omega_2}{\ddot{Y}_c} \left(\dot{\delta}_{2B} + \dot{\delta}_{1B} \right) = \sqrt{1+\varphi} \sqrt{2\xi - \xi^2} \tag{51}$$

Substituting these initial conditions into equation I. 6, in Appendix I, the motion after bottoming is described by the equation

$$\frac{\omega_2^2 \delta_2}{\ddot{Y}_c} = 1 - \sqrt{1 + \varphi \xi [2 - \xi]} \sin(\omega_2 t + \vartheta) \tag{52}$$

where $\sin \vartheta = \dfrac{-[1 - \xi]}{\sqrt{1 + \varphi \xi [2 - \xi]}}$ (53)

Obviously the maximum value of $\frac{\omega_2^2 \delta_2}{\ddot{Y}_c}$ is

$$\frac{\omega_2^2 \delta_{2_{MAX}}}{\ddot{Y}_c} = 1 + \sqrt{1 + \varphi \xi [2 - \xi]} \tag{54}$$

An interesting limit is provided by the case $\omega_1 \rightarrow 0$.

Here $$\frac{\omega_2^2 \delta_{2_{MAX}}}{\ddot{Y}_c} \rightarrow 1 + \sqrt{1 + \frac{2\omega_2^2 \delta_{1B}}{\ddot{Y}_c}} \tag{55}$$

This checks with other zero-zero cushion solutions.

For convenience let

$$\psi = \frac{\omega_2^2 \delta_{1B}}{\ddot{Y}_c} = \xi \left(\frac{\omega_2}{\omega_1}\right)^2 = \varphi \xi \tag{56}$$

Then $\xi = \psi / \varphi$

$$\frac{\omega_2^2 \delta_{2_{MAX}}}{\ddot{Y}_c} = 1 + \sqrt{1 + \psi [2 - \psi/\varphi]} \tag{57}$$

The critical value is

$$\psi_{CRIT} = 2\varphi \tag{58}$$

For $\psi > \psi_{CRIT}$ equation 57 is not applicable, and the solution is merely

$$\frac{\omega_2^2 \delta_{2_{MAX}}}{\ddot{Y}_c} = 2.0$$

The maximum value of equation 57 is found by differentiating with respect to ψ and equating to zero, holding ω_1^2 constant.

i e $$\frac{\partial}{\partial \psi}\left(\frac{\omega_2^2 \delta_{2_{MAX}}}{\ddot{Y}_c}\right) = \frac{(1 - \psi/\varphi)}{\sqrt{1 + \psi [2 - \psi/\varphi]}} = 0$$

$$\therefore \psi \Big]_{\delta_{MAX}} = \varphi \tag{59}$$

Substituting into equation I. 2, Appendix I

$$\left.\frac{\omega_2^2\,\delta_{2_{MAX}}}{\ddot{Y}_c}\right]_{MAX} \begin{array}{l} = 1+\sqrt{1+\varphi} \\ = 1+\sqrt{1+\psi} \end{array} \Bigg\} \tag{60}$$

Also, when the restraint stiffness is zero $\varphi \rightarrow \infty$ and

$$\left.\frac{\omega_2^2\,\delta_{2_{MAX}}}{\ddot{Y}_c}\right]_{\omega_1^2=0} \rightarrow 1+\sqrt{1+2\psi} \tag{61}$$

This is the "slack harness" (or zero cushion stiffness) solution.

(2) Impulsive velocity change Δv .

For an impulsive velocity change, we can regard the problem as a single degree of freedom. Prior to bottoming the effective spring stiffness is

$$\frac{1}{k_T} = \frac{1}{k_1} + \frac{1}{k_2} \tag{62}$$

$$\therefore \frac{1}{\Omega^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} \tag{63}$$

From Appe[illegible] I,

$$\delta_T = \frac{\Delta v}{\Omega} \sin \Omega t \tag{64}$$

For equality of force in the springs,

$$\delta_1 = \frac{k_2}{k_1}\,\delta_2$$

Put $$\delta_1 + \delta_2 = \delta_T = (1+\varphi)\,\delta_2$$

$$\therefore \delta_2 = \frac{\delta_T}{1+\varphi} \tag{65}$$

and $$\delta_1 = \frac{\varphi\, \delta_T}{1+\varphi} \tag{66}$$

Thus, from equations 64 and 66

$$\delta_{1B} = \frac{\Delta v}{\Omega} \cdot \frac{\varphi}{1+\varphi} \sin \Omega t_B \tag{67}$$

$$\delta_{2B} = \frac{\Delta v}{\Omega} \cdot \frac{\sin \Omega t_B}{1+\varphi} = \frac{\delta_{1B}}{\varphi} \tag{68}$$

$$\dot{\delta}_{2B} = \Delta v \frac{\cos \Omega t_B}{1+\varphi} \tag{69}$$

$$\dot{\delta}_{TB} = \Delta v \cos \Omega t_B \tag{70}$$

After bottoming, the spring stiffness is k_2 and the velocity of the mass with respect to the ground is $\dot{\delta}_T$. Thus, equation 64 becomes

$$\omega_2^2 \delta_2 = \frac{\omega_2^2 \delta_{1B}}{\varphi} \cos \omega_2 t + \omega_2 \Delta v \cos \Omega t_B \cdot \sin \omega_2 t \tag{71}$$

but $$\cos \Omega t_B = \sqrt{1 - \sin^2 \Omega t_B}$$

$$= \sqrt{1 - \left(\frac{\Omega \delta_{1B}}{\Delta v}\right)^2 \left(\frac{1+\varphi}{\varphi}\right)^2} \tag{72}$$

$$\therefore \quad \delta_{2\,MAX} = \frac{\Delta v}{\omega_2} \sqrt{1 - \left(\frac{\omega_2 \delta_{1B}}{\Delta v}\right) \frac{1}{\varphi}} \tag{73}$$

Since $$\frac{\omega_2^2}{\varphi} = \omega_1^2$$

and $$\omega_1^2 \delta_{1B} = \frac{k_1}{m} \delta_{1B}^2 = \frac{2E_c}{m}$$

$$\omega_2^2 \delta_{2\,MAX} = \omega_2 \Delta v \sqrt{1 - \frac{2E_c}{m \Delta v^2}} \tag{74}$$

where E_c = the energy storage parameter, being the total energy the resiliency can absorb at bottoming (ft. lb.)

m = mass of occupant (slugs)

The most favorable condition (δ_2 a minimum) is for the restraint system to just bottom out under the influence of an impulsive acceleration. This occurs when

$$\frac{\omega_2 \delta_{1B}}{\Delta v} = \frac{\varphi}{\sqrt{1+\varphi}}$$

or

$$\varphi_{OPT} = \frac{1}{2}\left(\frac{\omega_2 \delta_{1B}}{\Delta v}\right)^2 \left\{1 + \sqrt{1 + 4\left(\frac{\Delta v}{\omega_2 \delta_{1B}}\right)^2}\right\} \tag{75}$$

c. <u>"Crushable foam" (Constant force) restraint.</u>

This linear case may likewise be regarded as the simplest example of a nonlinear restraint system, and an explicit solution obtained.

We divide the system's behavior into three distinct regimes:

(A) when the spring force $k_2 \delta_2 < F$ the foam is "rigid."

(B) when $k_2 \delta_2 = F$ the foam deflects but the spring does not.

(C) when the foam bottoms out it becomes rigid again, and in addition to the acceleration input $\ddot{y}_c$, the dynamic model has to accommodate an impulsive velocity change $\dot{\delta}_{1B}$.

We now evaluate these regimes for a short period acceleration of magnitude $\ddot{Y}_c$ and zero rise time.

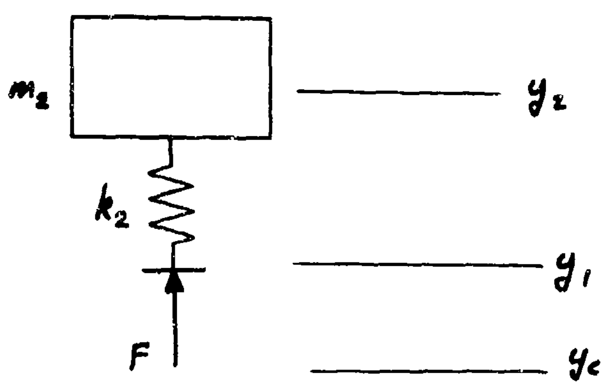


Figure 10. Dynamic model for a constant force restraint system.

(1) <u>Short period acceleration solution.</u>

<u>Condition (A)</u>

For this regime

$$\ddot{\delta}_2 + \omega_2^2 \delta_2 = \ddot{Y}_c \tag{76}$$

If the initial spring deflection is zero, the solution to this equation is

$$\frac{\omega_2^2 \delta_2}{\ddot{Y}_c} = 1 - \cos \omega_2 t \tag{77}$$

Obviously, the end of this regime is defined by

$$\frac{\omega_2^2 \delta_{2cR}}{\ddot{Y}_c} = \frac{F/m}{\ddot{Y}_c} = 1 - \cos \omega_2 t_{cR}$$

or

$$\cos \omega_2 t_{cR} = 1 - \frac{F/m}{\ddot{Y}_c} \tag{78}$$

The spring velocity $\dot{\delta}_{2cR}$ at this time is given by

$$\dot{\delta}_{2cR} = \frac{\ddot{Y}_c}{\omega_2} \sin \omega_2 t_{cR} = \frac{\ddot{Y}_c}{\omega} \sqrt{\frac{F/m}{\ddot{Y}_c}} \sqrt{2 - \frac{F/m}{\ddot{Y}_c}} \tag{79}$$

<u>Condition (B)</u>

The spring has now reached its critical deflection δ_{2cR} and remains at this value.

Now

$$\begin{aligned} \delta_1 &= \lambda_1 - (y_1 - y_c) \\ \delta_2 &= \lambda_2 - (y_2 - y_1) \end{aligned} \tag{80}$$

But $\ddot{y}_2 = F/m$ and $\ddot{\delta}_2 = 0$

$$\therefore \quad \ddot{y}_1 = \ddot{y}_2 + \ddot{\delta}_2 = F/m$$

$$\ddot{\delta}_1 = \ddot{y}_c - \ddot{y}_1 = \ddot{y}_c - F/m$$

$$\dot{\delta}_1 = \ddot{Y}_c t - (F/m)t + \dot{\delta}_{2cR} \tag{81}$$

$$\delta_1 = \tfrac{1}{2}t^2\left[\ddot{Y}_c - F/m\right] + \dot{\delta}_{2cR}\, t \tag{82}$$

Bottoming out occurs when $\delta_1 = \delta_{1B}$

i e, $t_B^2 + \dfrac{2\,\dot{\delta}_{2cR}}{\ddot{Y}_c - F/m}\, t_B - \dfrac{2\,\delta_{1B}}{\ddot{Y}_c - F/m}$

$$\therefore\ t_B = \frac{\dot{\delta}_{2cR}}{\ddot{Y}_c - F/m}\left\{-1 \pm \sqrt{1 + \frac{2\,\delta_{1B}}{\dot{\delta}^2_{2cR}}\left[\ddot{Y}_c - F/m\right]}\right. \tag{83}$$

Substituting equation 79 for $\dot{\delta}_{2cR}$ and defining $\dfrac{F/m}{\ddot{Y}_c} = \Phi$

$$\omega t_B = \frac{\sin\omega t_{cR}}{1-\Phi}\left\{-1 + \sqrt{1 + \frac{2\omega_2^2}{\ddot{Y}_c}\,\frac{[1-\Phi]}{\sin^2\omega t_{cR}}}\right. \tag{84}$$

Note that from equation 78

$$\cos\omega t_{cR} = 1 - \Phi$$

$$\sin\omega t_{cR} = \sqrt{2\Phi - \Phi^2}$$

$$\dot{\delta}_{2cR} = \frac{\ddot{Y}_c}{\omega}\sqrt{\Phi(2-\Phi)} \tag{85}$$

$$\therefore\ \omega t_B = \frac{\sqrt{\Phi(2-\Phi)}}{1-\Phi}\left\{-1 + \sqrt{1 + \frac{\omega_2^2\,\delta_{1B}}{\ddot{Y}_c}\cdot\frac{2(1-\Phi)}{\Phi(2-\Phi)}}\right. \tag{86}$$

From equation 81 the foam will bottom out with a velocity

$$\frac{\omega_2\,\dot{\delta}_{1B}}{\ddot{Y}_c} = \sqrt{\Phi(2-\Phi)}\sqrt{1 + \frac{\omega_2^2\,\delta_{1B}}{\ddot{Y}_c}\,\frac{2(1-\Phi)}{\Phi(2-\Phi)}} \tag{87}$$

Condition (C)

In condition (C) the equation of motion returns to the form of equation 76. The initial conditions are no longer zero, and the solution becomes

$$\frac{\omega^2 \delta_2}{\ddot{Y}_c} = 1 - \left(1 - \frac{\omega_2^2 \delta_{2a}}{\ddot{Y}_c}\right)\cos\omega_2 t + \frac{\omega_2 \dot{\delta}_{2B}}{\ddot{Y}_c}\sin\omega_2 t \tag{88}$$

Obviously,

$$\omega_2 \dot{\delta}_{2B} = \omega_2 \dot{\delta}_{1B} \tag{89}$$

Now $\dot{\delta}_2$ = 0 prior to bottoming, and the velocity change $\dot{\delta}_{1B}$ has been acquired impulsively by the spring at the instant of bottoming, thus substituting equation 86 into 88, and remembering that

$$\frac{\omega^2 \delta_2}{\ddot{Y}_c} = 1 - (1-\Phi)\cos\omega_2 t + \left[(1-\Phi)\omega t_B + \sqrt{\Phi(2-\Phi)}\,\sin\omega_2 t\right] \tag{90}$$

$$\left.\frac{\omega_2^2 \delta_2}{\ddot{Y}_c}\right]_{MAX} = 1 \pm (1-\Phi)\sqrt{(1+\omega_2^2 t_B^2) + \frac{2\sqrt{\Phi(2-\Phi)}\,\omega t_B}{1-\Phi} + \frac{\Phi(2-\Phi)}{(1-\Phi)^2}} \tag{91}$$

where ωt_B is given by equation 86.

If we define $\rho = \sqrt{\Phi(2-\Phi)}/(1-\Phi)$

$$\left.\frac{\omega_2^2 \delta_2}{\ddot{Y}_c}\right]_{MAX} = 1 + (1-\Phi)\sqrt{1 + (\omega t_B + \rho)^2} \tag{92}$$

$$\omega_2 t_B = \rho\left[-1 + \sqrt{1 + \frac{\omega^2 \delta_{1B}}{\ddot{Y}_c}\,\frac{2}{\rho^2}}\ \right] \tag{93}$$

Thus, the only variables are Φ and $\dfrac{\omega_2^2 \delta_{1B}}{\ddot{Y}_c}$

Some limit cases are of interest. For example, as $\Phi = \dfrac{F/m}{\ddot{Y}_c} \rightarrow$ 1.0,

$$\omega t_B \rightarrow \frac{\omega_2^2 \delta_{1B}}{\ddot{Y}_c}(1-\Phi) \qquad\qquad \frac{\omega_2^2 \delta_{2MAX}}{\ddot{Y}_c} \rightarrow 2.0$$

Similarly for $\Phi \rightarrow$ 0

$$\frac{\omega_2^2 \delta_{2MAX}}{\ddot{Y}_c} \rightarrow 1 + \sqrt{1 + 2\frac{\omega_2^2 \delta_{1B}}{\ddot{Y}_c}}$$

which is the same as equation 55.

(2) <u>Impulsive velocity change Δv.</u>

In this case the energy to be absorbed by the total system is $\frac{m}{2} \Delta v^2$. Before the foam starts to crush, the model will absorb

$$\Delta E_{(1)} = \frac{1}{2} k \, \delta_{(1)}^2 \tag{94}$$

If F is the crushing force, and δ_{1B} the bottoming depth, the energy absorbed in the foam will be $F\,\delta_{1B}$. Thus, the remaining energy to be absorbed by the spring will be

$$\Delta E_{(3)} = \frac{1}{2} m \, \Delta v^2 - F\,\delta_{1B} - \frac{1}{2} k \, \delta_{(1)}^2 \tag{95}$$

Dividing throughout by m,

$$\omega_2^2 \delta_{MAX}^2 = \Delta v^2 - 2(F/m)\,\delta_{1B} \tag{96}$$

$$\omega_2^2 \delta_{2MAX} = \sqrt{(\omega_2 \Delta v)^2 - 2\omega_2^2 \delta_{1B}(F/m)} \tag{97}$$

Naturally, if the initial energy $\frac{1}{2} m \Delta v^2$ is insufficient to bottom the foam, then

$$\omega_2^2 \delta_{2_{MAX}} = F/m$$

Also, if

$$\omega_2^2 \delta_{(1)} < F/m \tag{98}$$

The result will be

$$\omega_2^2 \delta_{MAX} = \omega_2 \Delta v \tag{99}$$

A convenient parameter is therefore

$$\frac{\omega_2^2 \delta_{2_{MAX}}}{\omega \Delta v} = \sqrt{1 - 2 \frac{(F/m)\, \delta_{1B}}{\Delta v}} \tag{100}$$

The optimum crushing force for a given velocity change is the one which results in the lowest value of $\omega_2^2 \delta_{2_{MAX}}$ for a given bottoming depth δ_{1B} .

This condition obviously occurs when the foam bottoms at the same instant that all the initial kinetic energy is absorbed. In other words, from equation 95

$$\frac{1}{2} m \Delta v^2 - F \delta_{1B} - \frac{1}{2} k_2 \delta_{(1)}^2 \tag{101}$$

But

$$\frac{1}{2} k_2 \delta_{(1)}^2 = \frac{1}{2} \frac{F^2}{k_2} \tag{102}$$

$$\therefore \omega_2^2 \delta_{(1)} = F/m \tag{103}$$

and from equations 101 and 102

$$F/m\Big)_{OPT} = -\omega_2^2 \delta_{1B} + \sqrt{(\omega_2^2 \delta_{1B})^2 + (\omega_2 \Delta v)^2} \tag{104}$$

and Dynamic Response Index (DRI)

$$\frac{\text{DRI with optimum restraint}}{\text{DRI with no restraint}} = -\frac{\omega_2 \delta_{16}}{\Delta v} + \sqrt{\left(\frac{\omega_2 \delta_{16}}{\Delta v}\right)^2 + 1} \tag{105}$$

d. The influence of slack in the restraint system.

When there is slack in a restraint system, or the occupant is separated from it, as in ejection from an aircraft subject to negative g, an initial velocity

$$\Delta v_o \doteq \int_0^{t_o} \ddot{y}_c \, dt \tag{106}$$

is built up prior to contact with the restraint system (which occurs at time $t = t_o$). Thus, the equations already developed are still valid, but the initial velocity condition also appears in the solution, increasing (most usually) the physiological effect of the applied acceleration.

The "slack distance" δ_s is related to the input acceleration by the equation

$$\delta_s = \int_0^{t_o} \Delta v \, dt = \iint_0^{t_o} \ddot{y}_c \, dt \, dt. \tag{107}$$

In analog work δ_s can be deterr .ined as a function of t_o for any input acceleration-time history. The value of Δv_o appropriate to the problem likewise can be determined and fed into the computer as an initial condition.

It is clear that if the acceleration input is an impulsive velocity change, restraint system slack will have no influence.

(1) The effect of slack when the restraint system is rigid.

For a rectangular (zero rise time) acceleration pulse ($\ddot{Y}_c$) in the short period regime, the velocity built up before contacting the rigid restraint system will be

$$\Delta v_o = \int_0^{t_o} \ddot{Y}_c \, dt = \ddot{Y}_c t_o \tag{108}$$

and the travel

$$\delta_s = \frac{1}{2} \ddot{Y}_c t^2 \tag{109}$$

Thus $$t_0 = \sqrt{2\delta_s/\ddot{Y}_c} \tag{110}$$

and $$\Delta v_0 = \sqrt{2\delta_s \ddot{Y}_c} \tag{111}$$

Substituting this initial condition in equation I. 7, Appendix I

$$\gamma = \frac{\omega \Delta v_0}{\ddot{Y}_c} = \omega\sqrt{2\delta_s/\ddot{Y}_c} \tag{112}$$

$$\frac{\omega^2 \delta}{\ddot{Y}_c} = 1 - \frac{e^{-\bar{c}\omega t}}{\eta}\sqrt{1 - 2\bar{c}\gamma + \gamma^2}\ \sin(\omega t + \chi) \tag{113}$$

where

$$\sin \chi = \eta\Big/\sqrt{1 - 2\bar{c}\gamma + \gamma^2} \tag{114}$$

We now have to find the solution for the maximum value of equation 113 which can be done using the techniques of section I. 3 in Appendix I. A much simpler solution can be obtained for zero damping, of course, since equations 113 and 114 then become

$$\left.\begin{aligned} \frac{\omega^2 \delta}{\ddot{Y}_c} &= 1 - \sqrt{1 + \frac{2\omega^2 \delta_s}{\ddot{Y}_c}}\ \sin\{\omega t + \chi\} \\ \sin \chi &= 1\Big/\sqrt{1 + \frac{2\omega^2\delta_s}{\ddot{Y}_c}} \end{aligned}\right\} \tag{115}$$

Obviously $$\frac{\omega^2 \delta_{MAX}}{\ddot{Y}_c} = 1 + \sqrt{1 + \frac{2\omega^2\delta_s}{\ddot{Y}_c}} \tag{116}$$

is a limit solution and has been obtained elsewhere in this report. Equation 116 is plotted in figure 11. It is interesting to note that a slack of only half an inch (δ_s = .04167) in the spinal mode (ω = 251.0 rads/sec) would increase the DRI of a 20 g acceleration pulse (with zero rise time) by as much as 100%.

(2) <u>The effect of slack in a linear spring restraint system.</u>

The equation of motion for this case is given in Section 3.b. When damping in the dynamic model is neglected equation 45 gives the solution

$$\frac{\omega_2^2 \delta_2}{\ddot{Y}_c} = 1 - \left\{ \frac{\cos \omega_2 t}{\sqrt{1+\varphi}} - \frac{\omega_2}{\sqrt{1+\varphi}}\sqrt{\frac{2\delta_s}{\ddot{Y}_c}}\ \sin \frac{\omega_2 t}{\sqrt{1+\varphi}} \right\} \tag{117}$$

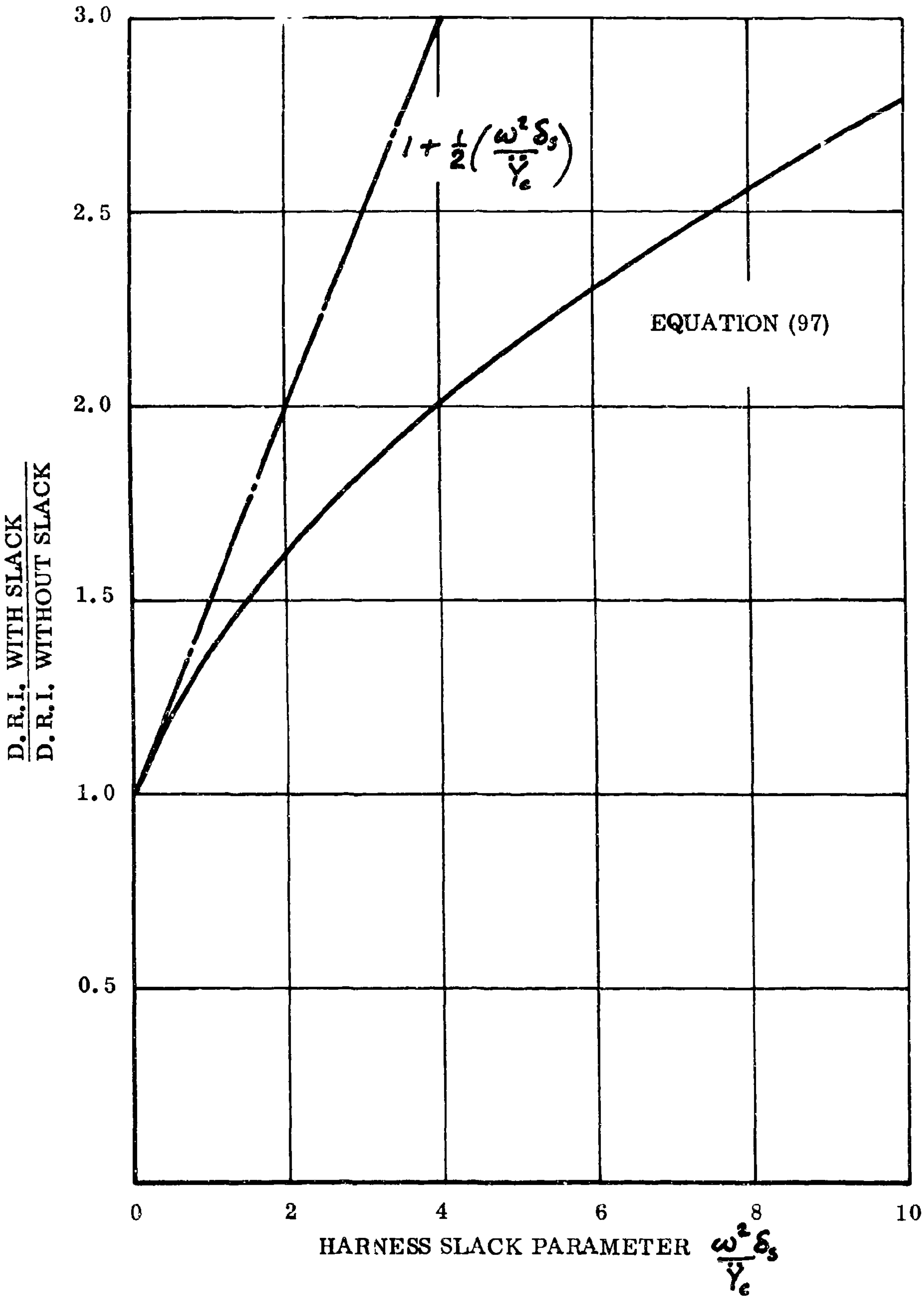


Figure 11. Increase in D. R. I. due to slack when the restraint system is rigid, for a zero rise time pulse. (Zero damping dynamic model).

$$\frac{\omega_2^2 \delta_2}{\ddot{Y}_c} = 1 - \sqrt{(1+\varphi) + \omega_2^2 \frac{2\delta_s}{\ddot{Y}_c}} \cos\left[\frac{\omega_2 t}{\sqrt{1+\varphi}} - \theta_1\right] \tag{118}$$

where $\cos\theta_1 = \sqrt{(1+\varphi)} \Big/ \sqrt{(1+\varphi) + \omega_2^2 \cdot \frac{2\delta_s}{\ddot{Y}_c}}$

After the spring has bottomed we have, from equation 51

$$\frac{\omega_2^2 (\dot{\delta})_B}{\ddot{Y}_c} = \frac{\omega^2}{\ddot{Y}_c}(\dot{\delta}_{2B} + \dot{\delta}_{1B}) \tag{119}$$

$$\frac{\omega_2^2 \delta_2}{\ddot{Y}_c} = 1 - (1-\xi)\cos\omega_2 t + \frac{\omega_2 (\dot{\delta})_B}{\ddot{Y}_c} \sin\omega_2 t$$

From equation 117

$$\frac{\omega_2 \dot{\delta}_{2B}}{\ddot{Y}_c} = \frac{\omega_2}{\sqrt{1+\varphi}} \sin\frac{\omega_2 t_B}{\sqrt{1+\varphi}} + \frac{\omega_2^2}{1+\varphi}\sqrt{\frac{2\delta_s}{\ddot{Y}_c}} \cos\frac{\omega_2 t_B}{\sqrt{1+\varphi}}$$

and since

$$\delta_{2B} = \frac{\delta_{1B}}{\varphi}$$

$$\frac{\omega_2 (\dot{\delta})_B}{\ddot{Y}_c} = \omega_2 \sqrt{(1+\varphi) + \omega_2^2 \cdot \frac{2\delta_s}{\ddot{Y}_c}} \sin\left[\frac{\omega_2 t_B}{\sqrt{1+\varphi}} + \theta_2\right] \tag{120}$$

where $\cos\theta_2 = \sqrt{1+\varphi} \Big/ \sqrt{(1+\varphi) + \omega_2^2 \cdot \frac{2\delta_s}{\ddot{Y}_c}}$

From equation 118

$$\cos\left[\frac{\omega_2 t_B}{\sqrt{1+\varphi}} - \theta_1\right] = \frac{\left[1 - \frac{\omega_1^2 \delta_{1B}}{\ddot{Y}_c}\right]\sqrt{1+\varphi}}{\sqrt{(1+\varphi) + \omega_2^2 \frac{2\delta_s}{\ddot{Y}_c}}} \tag{121}$$

Obviously $\theta_1 = \theta_2$, but the different signs in equations 120 and 121 prevent us from substituting for $(\theta + \omega_2 t_B/\sqrt{1+\varphi})$ in equation 120. Thus, a single explicit relationship cannot be obtained.

By substituting equation 120 into 119 we have

$$\frac{\omega_2^2 \delta_{2max}}{\ddot{Y}_c} = 1 + \sqrt{(1+\xi)^2 + \omega_2^2\left[(1+\varphi) + \frac{\omega_2^2\, 2S_s}{\ddot{Y}_c}\right] \sin^2\left[\frac{\omega_2 t_6}{\sqrt{1+\varphi}} + \theta_3\right]} \tag{122}$$

where

$$\cos\theta_3 = \frac{\sqrt{1+\varphi}}{\sqrt{(1+\varphi) + \frac{\omega_2^2\, 2S_s}{\ddot{Y}_c}}}$$

and t_6 is given by equation 121.

e. The influence of preloading in a restraint system.

The effects of preloading can be examined effectively with an analog computer, where preloading can be simulated as an initial condition acceleration input $\ddot{Y}_{c_0}$.

For the simple case of a rigid restraint system, however, subjected to a zero rise time acceleration we can obtain a solution from the work in Appendix I. From equation I.7, $\varphi = 0$, so that $\lambda t_{max} = n\pi$. Thus, equation I.14 becomes

$$\frac{\omega_2^2 \delta_{2max}}{\ddot{Y}_c} = 1 + e^{\frac{-\pi\bar{c}}{\eta}}\left(1 - \frac{\omega^2 \delta_{20}}{\ddot{Y}_c}\right) \tag{123}$$

In figure 12 the dynamic overshoot for zero preload is plotted, and in figure 13 the influence of preload is included. It was found in the last section that the parameter influencing the physiological effect of a zero-rise time acceleration $\ddot{Y}_c$ was

$$\frac{\omega_2^2 S_s}{\ddot{Y}_c}$$

where δ_s is the slack distance. For preloading the parameter is

$$\frac{\omega_2^2 \delta_{20}}{\ddot{Y}_c} = \frac{\text{preload force in lb, resolved along the appropriate axis}}{\text{weight of occupant in lb x acceleration in g's}}$$

As an example, consider the spinal mode which is of importance in aircraft ejection seat work and for which $\bar{c}$ = 0.224. The maximum tolerable preload, even with a powered inertia reel, is probably about 1 g, resolved along the spinal axis, so that

$$\frac{\omega_2^2 \delta_{20}}{\ddot{Y}_c} = \frac{1}{\ddot{Y}_c}$$

For $\ddot{Y}_c$ = 15 g say, $\frac{\omega^2 \delta_{20}}{\ddot{Y}_c}$ = .0667

Thus, the attenuation factor, from figure 13, is about 0.98, or a 2% reduction in DRI.

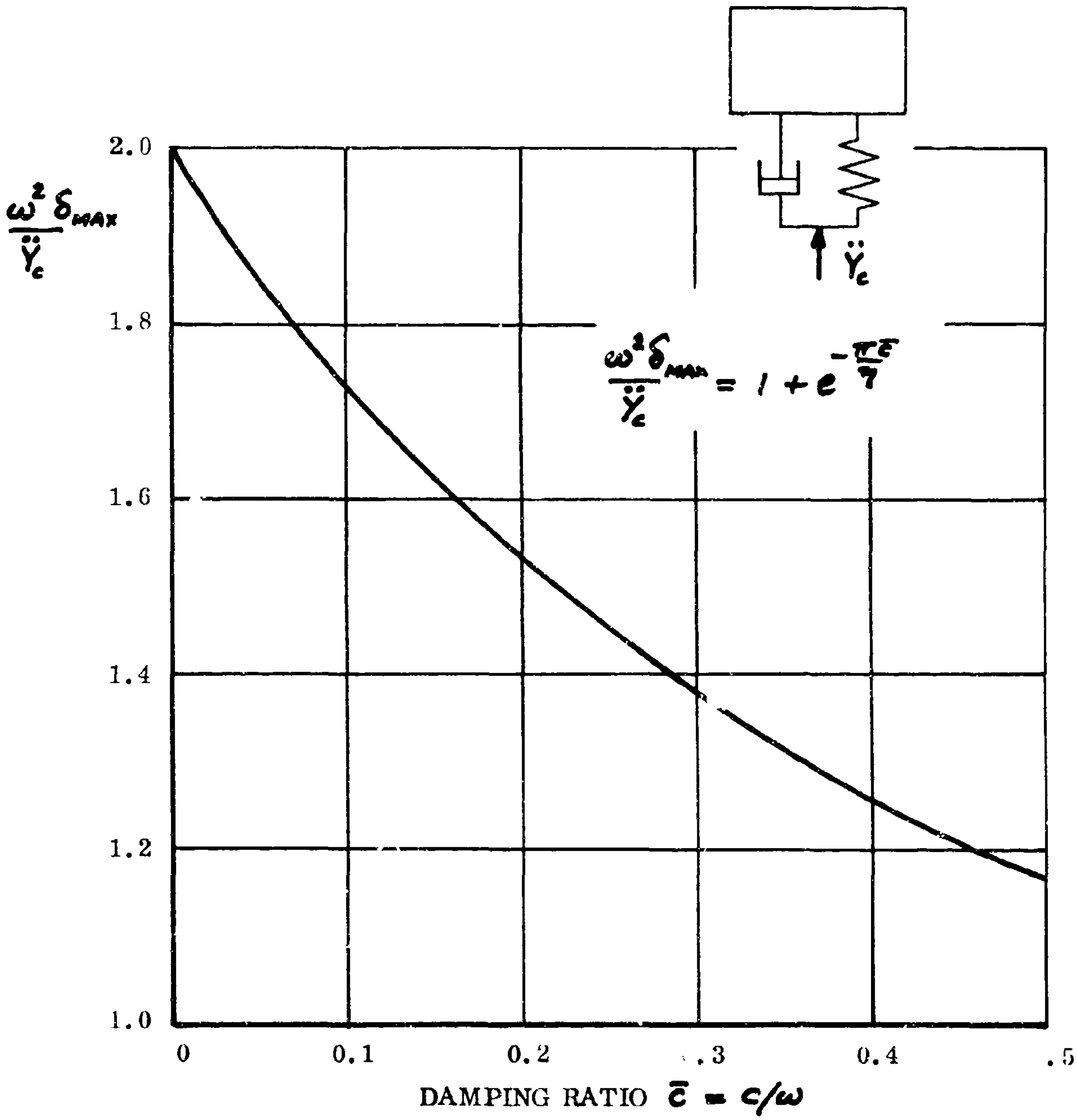
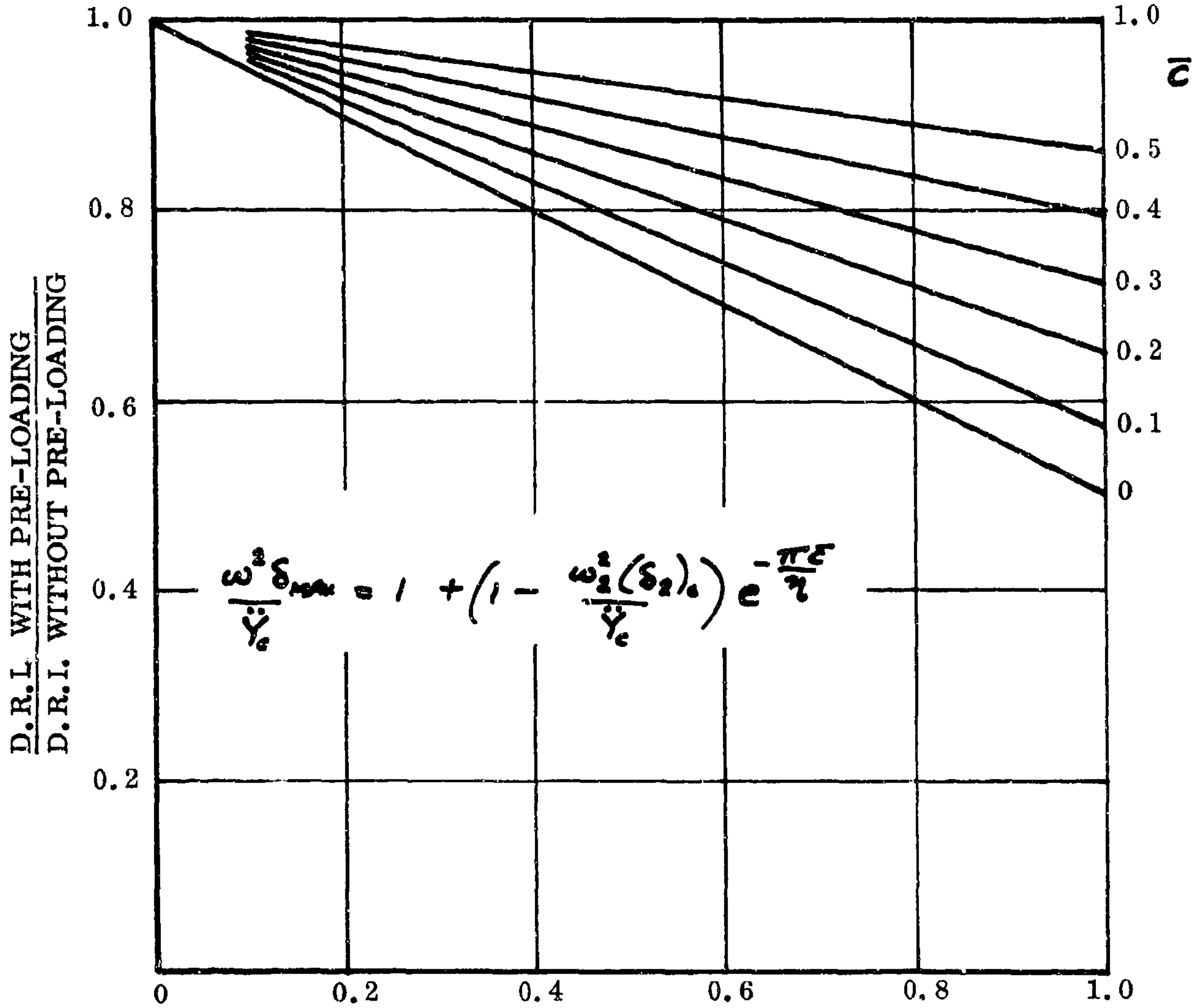


Figure 12. Dynamic overshoot in response to a zero rise time acceleration $\ddot{Y}_c$ for a damped linear system, with zero restraint system preload.



RESTRAINT PRE-LOAD PARAMETER =

$$\frac{\omega_2^2 (\delta_2)_0}{\ddot{Y}_c} = \frac{\text{PRE-LOAD FORCE (LB)}}{\text{WT. OF OCCUPANT (LB) X ACCELERATION (g's)}}$$

Figure 13. Influence of preloading restraint system on $\omega^2 \bar{\delta}_{max}$. (Damped linear system subjected to a zero rise time acceleration input $\ddot{Y}_c$).

It is obvious from the foregoing that preloading the restraint system has little value, from a physiological point of view. Since it is of extreme importance to avoid slack, however, as shown in the preceeding section, some preloading should always be introduced in the process of cinching down the restraint system to make sure that no slack exists.

4. SOME SIMPLE NONLINEAR RESTRAINT CASES INVOLVING CONTINUOUS FUNCTIONS

The simplest type of nonlinear restraint problem consists of a zero-damping model of the human body in series with an undamped, nonlinear spring. Thus, the problem may be regarded as a nonlinear extension of the work in Section II. 3. b., page 18.

From equation 14 the appropriate equations of motion will be

$$\left.\begin{aligned} \ddot{\delta}_2 + F_2 &= \ddot{y}_c - \ddot{\delta}_1 \\ F_1 &= F_2 \end{aligned}\right\} \tag{124}$$

or, more specifically

$$\left.\begin{aligned} \ddot{\delta}_2 + \omega_2^2 \delta_2 &= \ddot{y}_c - \ddot{\delta}_1 \\ \omega_2^2 \delta_2 &= f_s(\delta_1) \end{aligned}\right\} \tag{125}$$

where $f_s(\delta_1)$ is some continuous function. By differentiating the $f_s(\delta_1)$ equation we can obtain an expression for $\ddot{\delta}_1$ which in some (very few) cases enables a general solution to be obtained.

Retaining the generality of equation 125

$$\omega_2^2 \dot{\delta}_2 = \frac{d}{d\delta_1}[f_s(\delta_1)]\,\dot{\delta}_1 \tag{126}$$

$$\omega_2^2 \ddot{\delta}_2 = \frac{d}{d\delta_1}[f_s(\delta_1)]\,\ddot{\delta}_1 + (\dot{\delta}_1)^2 \frac{d^2}{d\delta_1^2}[f_s(\delta_1)] \tag{127}$$

Thus,

$$\ddot{\delta}_1 = \frac{\omega_2^2 \ddot{\delta}_2}{\frac{d}{d\delta_1}[f_s(\delta_1)]} - \frac{(\dot{\delta}_1)^2 \frac{d^2}{d\delta_1^2}[f_s(\delta_1)]}{\frac{d}{d\delta_1}[f_s(\delta_1)]} \tag{128}$$

Substituting for $\dot{\delta}_1$, from equation 126 and denoting differentiation with respect to δ_1, by a prime;

$$\ddot{\delta}_1 = \frac{\omega_2^2 \ddot{\delta}_2}{f_s'(\delta_1)} - \frac{f_s''(\delta_1)\,\omega_2^4(\dot{\delta}_2)^2}{[f_s'(\delta_1)]^3} \tag{129}$$

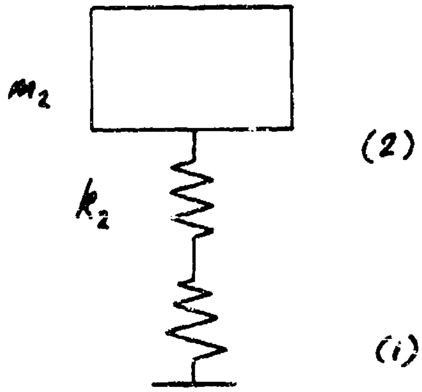


Figure 14. Simple model of a nonlinear restraint system.

Substituting into equation 125

$$\ddot{\delta}_2\left[1+\frac{\omega_2^2}{f_\varsigma'(\delta_1)}\right]-\frac{f_\varsigma''(\delta_1)\,\omega_2^4}{[f_\varsigma'(\delta_1)]^3}\left(\dot{\delta}_2\right)^2+\omega_2^2\,\delta_2=\ddot{y}_c \tag{130}$$

Even though we have succeeded in reducing the problem to a single degree of freedom, this equation is difficult to solve, whatever the nonlinear function $f_\varsigma\ (\delta_1)$ may be. The only cases which can be solved this way are of little value in practical work.

As an alternative approach, we can consider the two springs in figure 14 as being one spring with a rather more complex force-deflection relationship. This approach is more promising because the solution for a single nonlinear spring-mass problem can be fairly simple.

Retaining our generality, and defining the total deflection of the two springs as

$$\delta_T=f_\varsigma^{-1}\left(\frac{F}{m}\right)+\frac{F}{m}\cdot\frac{1}{\omega_2^2} \tag{131}$$

If this can be solved for F/m in terms of δ_T, then the resulting equation describes the equivalent single nonlinear spring.

For the general nonlinear restraint spring examined in the next section, equation 131 takes the following form

$$F/m=f_\varsigma(\delta_1)=\varsigma_n\,\delta_1^{\,n} \tag{132}$$

$$\delta_T=\left(\frac{F}{m}\right)\frac{1}{\omega_2^2}+\frac{1}{\varsigma_n^{1/n}}\left(\frac{F}{m}\right)^{\frac{1}{n}} \tag{133}$$

and $$\frac{F}{m} = \xi_n \left[\delta_T^{\,n} \cdot\cdot\, n\left(\frac{F}{m}\right)\frac{\delta_T^{\,n-1}}{\omega_2^{\,2}} + \frac{n(n-1)}{2!}\left(\frac{F}{m}\right)^2 \frac{\delta_T^{\,n-2}}{\omega_2^{\,4}} - \frac{n(n-1)(n-2)}{3!}\left(\frac{F}{m}\right)^3 \frac{\delta_T^{\,n-3}}{\omega_2^{\,6}} + \cdots \right] \tag{134}$$

In order to evolve an explicit expression, we must solve this polynominal in (F/m), which is easy to do for n = 1 or 2. For n = 3 or 4 the solutions become increasingly complicated, and higher powers become impractical.

A closed form solution is possible for the simplest polynominal nonlinearity, which is

$$\frac{F}{m} = f_\xi(\delta_1) = \omega_1^{\,2}\delta_1 + \alpha\,\delta_1^{\,2} \tag{135}$$

Since $\delta_1 = \delta_T - \delta_2$

$$\frac{F}{m} = \omega_1^{\,2}(\delta_T - \delta_2) + \alpha\,(\delta_T - \delta_2)^2 \tag{136}$$

and the solution for (F/m) is found to be

$$\frac{F}{m} = \frac{\omega_2^{\,4}}{2\alpha}\left[\left(1 + \left(\frac{\omega_1}{\omega_2}\right)^2 + 2\frac{\alpha\,\delta_T}{\omega_2^{\,2}}\right) - \sqrt{\left[1 + \left(\frac{\omega_1}{\omega_2}\right)^2\right]^2 + \frac{4\alpha\,\delta_T}{\omega_2^{\,2}}}\,\right] \tag{137}$$

The same type of solution could also be obtained for the cubic nonlinearity

$$\frac{F}{m} = \omega_1^{\,2}\delta + \beta\,\delta_1^{\,3} \tag{138}$$

but with added complexity.

Another type of nonlinearity considered in the next section is the tangent relationship

$$\frac{F}{m} = \frac{2\omega_0^{\,2}}{\pi}\,\delta_B \tan\frac{\pi\delta}{2\delta_B} \tag{139}$$

$$= \frac{2\omega_0^{\,2}}{\pi}\,\delta_{1B} \tan\frac{\pi}{2\delta_{1B}}\left(\delta_T - \frac{F}{m\,\omega_2^{\,2}}\right) \tag{140}$$

Obviously no explicit expression for (F/m) can be obtained in this case.

From the foregoing it is evident that, even for the simplest possible zero damping cases, it is generally not possible to achieve closed form solutions for the effects of a nonlinear restraint system, except for the quadratic case. Consequently, the only practical method of treatment is to employ an analog or digital computer.

a. Generalized power law nonlinearity.

When the spring force-deflection curve is described by the equation

$$\frac{F}{m} = \xi_n \delta^n \tag{141}$$

the equation of motion of a simple system becomes

$$\ddot{\delta} + \xi_n \delta^n = \ddot{y}_c \tag{142}$$

(1) Solution for an impulsive velocity change Δv.

The potential energy stored in the spring at a deflection δ_{max} is

$$E = m \int_{\delta_0}^{\delta_{max}} (F/m)\, d\delta \tag{143}$$

$$\frac{E}{m} = \xi_n \int_{\delta_0}^{\delta_{max}} \delta^n \, d\delta = \frac{\xi_n}{n+1} \left[\delta_{max}^{n+1} - \delta_0^{n+1} \right] \tag{144}$$

Equating 144 to the initial kinetic energy $\frac{1}{2} m \Delta v^2$ before impact,

$$\Delta v^2 = \frac{2 \xi_n}{n+1} \left[\delta_{max}^{n+1} - \delta_0^{n+1} \right)$$

$$\delta_{max} = \left[\frac{n+1}{2} \frac{\Delta v^2}{\xi_n} + \delta_0^{n+1} \right]^{\frac{1}{n+1}} \tag{145}$$

For zero initial conditions, the DRI is

$$\xi_n \delta_{max}^n = \xi_n^{\frac{1}{n+1}} \Delta v^{\frac{2n}{n+1}} \left(\frac{n+1}{2} \right)^{\frac{n}{n+1}} \tag{146}$$

(2) <u>Solution for constant short-period acceleration $\ddot{Y}_c$.</u>

Let $\dot{\delta} = p$ in equation 142

Then $\ddot{\delta} = p \dfrac{dp}{d\delta}$

$$\int_0^{\dot{}} p\,dp = \int_0^{\delta_{MAX}} \left[\ddot{Y}_c - g_n \delta^n \right) d\delta = 0$$

$$\ddot{Y}_c (\delta_{MAX} - \delta_0) - \frac{g_n}{n+1} \left(\delta_{MAX}^{n+1} - \delta_0^{n+1} \right) = 0 \tag{147}$$

For zero initial conditions $\ddot{Y}_c = \dfrac{g_n}{n+1} \delta_{MAX}^n$

and the DRI is

$$g_n \delta_{MAX}^n = (n+1) \ddot{Y}_c \tag{148}$$

b. <u>Generalized cubic non-linearity.</u>

In this seotion we are concerned with an undamped model whose spring force-deflection curve is described by the equation

$$\frac{F}{m} = \omega^2 \delta + \beta \delta^3 \tag{149}$$

so that the equation of motion becomes

$$\ddot{\delta} + \omega^2 \delta + \beta \delta^3 = \ddot{y}_c \tag{150}$$

(1) <u>Solution for an impulsive velocity change Δv .</u>

The potential energy stored in the spring at a deflection δ_{MAX} is

$$E = m \int_{\delta_0}^{\delta_{MAX}} (F/m)\, d\delta \quad = m \int_{\delta_0}^{\delta_{MAX}} (\omega^2 \delta + \beta \delta^3)\, d\delta \tag{151}$$

$$\frac{E}{m} = \frac{1}{2} \left[\omega^2 (\delta_{MAX}^2 - \delta_0^2) + \frac{1}{2} \beta (\delta_{MAX}^4 - \delta_0^4) \right] \tag{152}$$

Equating 152 to the initial kinetic energy $\frac{1}{2} m \Delta v^2$ before impact

$$\Delta v^2 = \omega^2 (\delta_{MAX}^2 - \delta_0^2) + \frac{1}{2}\beta (\delta_{MAX}^4 - \delta_0^4) \tag{153}$$

For zero initial conditions

$$\frac{1}{2}\beta (\delta_{MAX}^2)^2 + \omega^2 (\delta_{MAX}^2) - \Delta v^2 = 0 \tag{154}$$

$$\delta_{MAX}^2 = \frac{-\omega^2 \pm \sqrt{\omega^4 + 2\beta \Delta v^2}}{\beta} \tag{155}$$

The DRI is obtained by substituting for δ_{MAX} in equation 149.

(2) Solution for constant short-period acceleration $\ddot{Y}_c$.

Let $\dot{\delta} = p$ in equation 150

Then $\ddot{\delta} = p \frac{dp}{d\delta}$

$$\int_0^{\dot{p}} p \, dp = \int_{\delta_0}^{\delta_{MAX}} [\ddot{Y}_c - \omega^2 \delta - \beta \delta^3] \, d\delta$$

$$\ddot{Y}_c (\delta_{MAX} - \delta_0) - \frac{\omega^2}{2} (\delta_{MAX}^2 - \delta_0^2) - \frac{\beta}{4} (\delta_{MAX}^4 - \delta_0^4) = 0 \tag{156}$$

For zero initial conditions

$$\ddot{Y}_c - \frac{1}{2}\omega^2 \delta_{MAX} - \frac{1}{4}\beta \delta_{MAX}^3 = 0 \tag{157}$$

δ_{MAX} is given by the solution to this cubic.

c. Generalized tangent law nonlinearity.

For this case, we are concerned with a spring whose nonlinear characteristic is as defined in figure 15.

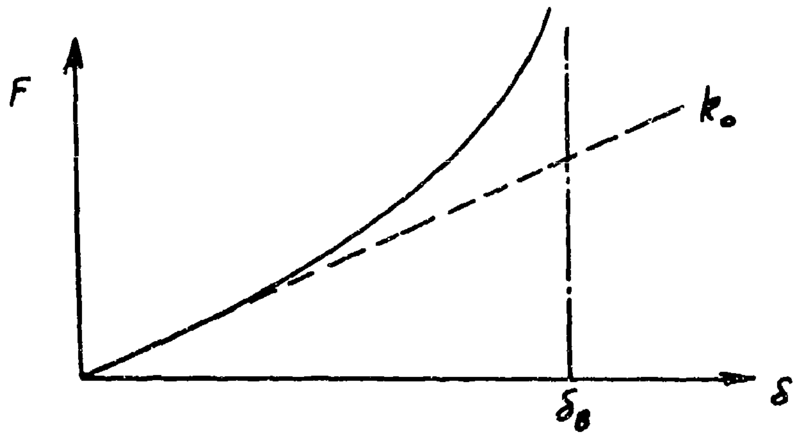


Figure 15. Tangent spring force deflection curve.

i e
$$F = \frac{2k_0\delta_B}{\pi}\tan\frac{\pi\delta}{2\delta_B} \tag{158}$$

Writing
$$\xi = \frac{\pi}{2\delta_B} \tag{159}$$

the equation of motion is
$$\ddot{\delta} + \frac{k_0}{m\xi}\tan\xi\delta = \ddot{y}_c \tag{160}$$

or writing $\frac{k_0}{m} = \omega_0^2$

$$\ddot{\delta} + \frac{\omega_0^2}{\xi}\tan\xi\delta = \ddot{y}_c \tag{161}$$

(1) <u>Solution for an impulsive velocity change Δv.</u>

The potential energy stored in the spring at a deflection δ_{MAX} is

$$\therefore\ E = \frac{m}{\xi}\omega_0^2\int_{\delta_0}^{\delta_{MAX}}\tan\xi\delta\,\frac{d\delta}{\xi}$$

$$= -\frac{m\omega_0^2}{\xi^2}\log\left|\frac{\cos\xi\delta_{MAX}}{\cos\xi\delta_0}\right| \tag{162}$$

Equating 162 to the initial kinetic energy $\frac{1}{2}m\Delta v^2$ before impact

$$\Delta v^2 = -\frac{2\omega_0^2}{\xi^2}\log\left|\frac{\cos\xi\delta_{MAX}}{\cos\xi\delta_0}\right| \tag{163}$$

for zero initial conditions

$$\log|\cos\xi\delta_{MAX}| = -\xi^2\frac{\Delta v^2}{2\omega_0^2}$$

$$\cos\xi\delta_{MAX} = e^{-\frac{1}{2}\left(\frac{\xi\Delta v}{\omega_0}\right)^2} \tag{164}$$

and the DRI is

$$\frac{F_{MAX}}{m} = \frac{\omega_0^2}{\xi}\tan\xi\delta_{MAX} \tag{165}$$

Now $\tan x = \sqrt{\sec^2 x - 1}$

$$\therefore \frac{F_{MAX}}{m} = \frac{\omega_0^2}{\xi}\sqrt{e^{\left(\frac{\xi\Delta v}{\omega_0}\right)} - 1} \tag{166}$$

(2) <u>Solution for constant short period acceleration $\ddot{Y}_c$.</u>

Let $\dot{\delta} = p$ in equation 161

Then $\ddot{\delta} = p\frac{dp}{d\delta} = \xi p\frac{dp}{d(\xi\delta)}$

$$\therefore \int_0^0 p\,dp = \int_{\xi\delta_0}^{\xi\delta_{MAX}}\left[\frac{\ddot{Y}_c}{\xi} - \frac{\omega_0^2}{\xi^2}\tan(\xi\delta)\right]d(\xi\delta) \tag{167}$$

$$\frac{\ddot{Y}_c}{\xi}(\xi\delta_{MAX} - \xi\delta_0) + \frac{\omega_0^2}{\xi^2}\left[\log|\cos\xi\delta_{MAX}| - \log|\cos\xi\delta_0|\right] = 0 \tag{168}$$

For zero initial conditions

$$\ddot{Y}_c = \frac{-\omega_0^2}{\xi^2\delta_{MAX}}\log|\cos\xi\delta_{MAX}| \tag{169}$$

There is no explicit solution for the DRI in this case.

SECTION III

THE DERIVATION OF DYNAMIC MODELS OF THE HUMAN BODY

1. GENERAL OBSERVATIONS

The basic aim in Body Dynamics is to derive a system of differential equations which are in harmony with the observed behavior of the human body when it is subjected to acceleration. Since there is always, or nearly always, a physical analog to a differential equation, it is frequently easier to discuss the analog rather than the equations it illustrates because our thinking is typically better adapted to considering physical objects than to reasoning in terms of mathematical abstractions.

A typical "dynamic model" is illustrated in figure 16. This analog has two degrees of freedom because the masses can move in an arbitrary manner with respect to each other.

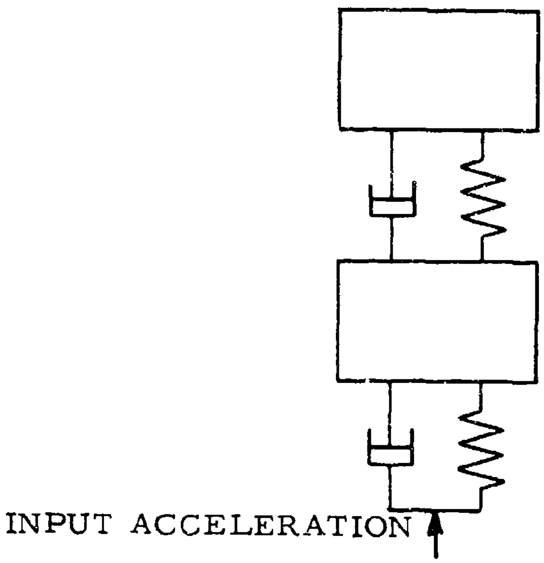


Figure 16. Dynamic analog or model for a two degree of freedom problem.

The model used in figure 16 is a "lumped parameter" model. That is, the mass, spring and damping characteristics are all considered as separate and discrete elements. Such treatment is not always possible. Dynamic models of the head, developed to predict head or brain injury, are based on the assumption that mass, damping and stiffness are distributed uniformly throughout the skull. As might be expected, the equations for this case are rather more complicated than for an equivalent lumped parameter model, but there is always a lumped parameter equivalent of such a system.

Dynamic models of the human body are based mainly upon experimental measurements of its response to various types of acceleration inputs as well as the subjective reactions of the individuals to the tests. Experiments with cadavers are also of value, although to a lesser extent since substantial changes occur in some of the body's dynamic characteristics after death.

Perhaps the most cogent reason for the use of dynamic models is that they provide an essentially unifying theory to all physical observations of human response to acceleration, thereby, permitting impact data, aircraft ejection seat firings, sled tests, shipboard injuries, and a host of other observations to be reduced to a common denominator, known as the "Dynamic Response Index" (DRI). A theory which enables all experimental observations to be reduced in terms of a single parameter can also be used in reverse, of course, to determine the probable results of an experiment which has yet to be carried out; or, putting it another way, to determine whether the experiment is worth carrying out at all.

In the present context, we are concerned with determining the numerical constants of lumped parameter dynamic models which will best represent the human body. We use as the basis of our effort the available experimental data.

Experimental data can be divided into three classes;

(a) Experiments in which the acceleration input pulse is of a type which enables the data to be used directly in determining the dynamic model parameters. The best example in this class is an impulsive or "Impact" velocity change.

(b) Cases in which the data must be reduced by means of a dynamic model before it can be used. An irregular ejection seat acceleration-time history is an example of this second class.

(c) Cases for which some or all of the acceleration records are worthless, but for which an appropriate idealization of the acceleration input can be deduced indirectly.

In general, it is easiest to use type (a) data in the generation of a dynamic model, and then to use this model to improve probability of injury estimates with type (b) and (c) data.

There are four different types of information which are of value in determining the values of a dynamic model for any particular direction.

From acceleration experiments we can determine:

(a) the critical impact velocity change Δv ,

(b) the critical short period acceleration $\ddot{y}_c$, and

(c) the "free oscillation" frequency ω_f (if the subject is unrestrained in rebound).

From impedance measurements we can determine:

(d) the "small amplitude" resonant frequency ω_{fo} .

2. METHODS OF DETERMINING LUMPED PARAMETER MODELS FROM THE AVAILABLE DATA.

a. Analysis of acceleration tolerance data alone.

Experiments in which an approximately rectangular acceleration pulse (figure 17) is imposed on a live human subject should result in injury (or more precisely, probability of injury) curves of the type shown in figure 18.

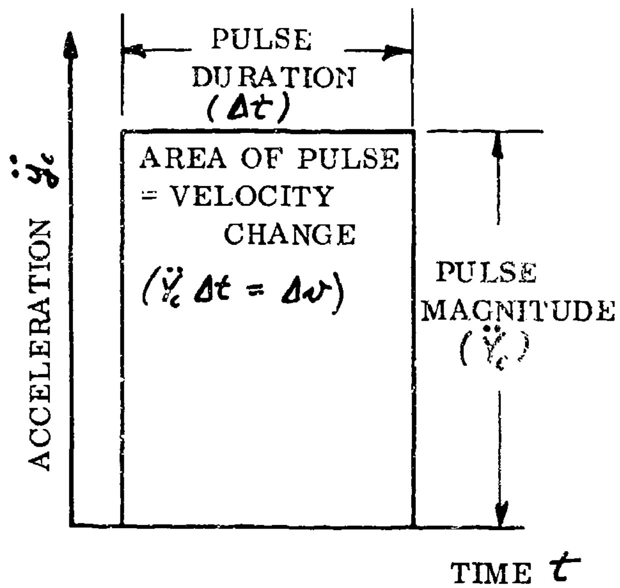


Figure 17. Definition of rectangular acceleration pulse.

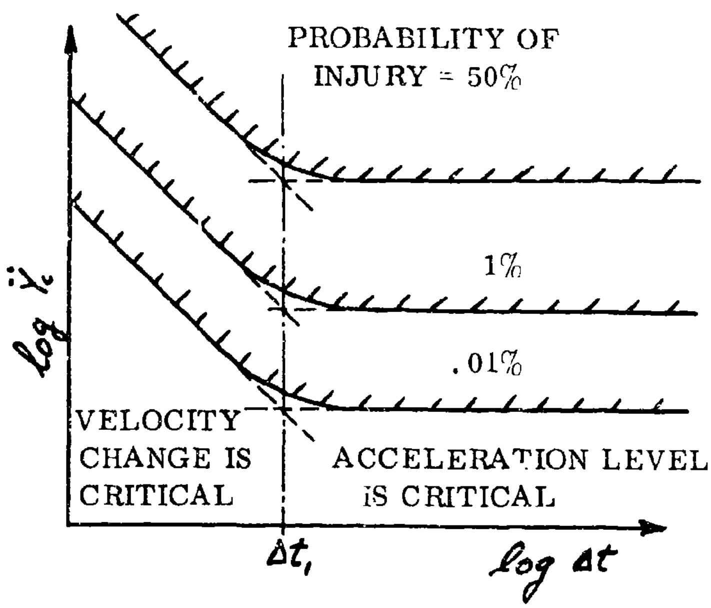


Figure 18. Typical injury curves for rectangular acceleration pulse.

The "corner duration" (Δt_1) marks the boundary between two dynamic regimes. For $\Delta t < \Delta t_1$ we are in the impact or impulsive velocity change regime. The pulse shape has no influence upon the behavior of the system, but only the pulse area, which is equal to the velocity change imposed. In practical terms, the pulse duration in this regime is so small that it is over before the body has started to respond. Thus, the body sees it merely as a change in velocity.

For $\Delta t > \Delta t_1$ the pulse shape is of major importance and, for a rectangular pulse, the peak forces generated in the body are a function only of the pulse magnitude ($\ddot{Y}_c$), irrespective of pulse length.

(1) Linear model with damping.

So far the discussion has been confined to modes in which the human body behaves as a "single degree of freedom" dynamic system, of the type shown in figure 19. In practice, this happens to be a good approximation of the human body's response.

In the case of a sitting subject experiencing a positive spinal (eyeballs down) acceleration, for example, we might give physical reality to figure 19 by the following identifications.

> The mass corresponds to the mass of the head and upper torso.
>
> The spring corresponds to the subject's spine.
>
> The damper corresponds to the distributed damping in the spine and associated tissues.

In Reference 10 Stech showed that, if the effective spring rate (k) was obtained from Cadaver data, this postulate could be used to calculate the frequency of a dynamic model representing the human body. The result agreed with other measurements made with live human subjects. However, it is important to realize that such identifications cannot always be made (the transverse case for example) and that, strictly speaking, they are not necessary to our purpose.

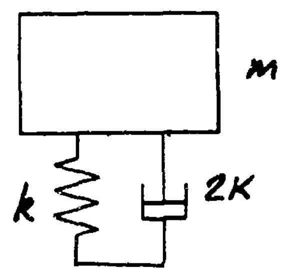


Figure 19. Single degree of freedom, linear dynamic system.

We shall now consider the mathematics of a single degree of freedom system in order to determine the lumped parameter coefficients for particular tolerance curves of the type shown in figure 18, when the dynamic model is linear. The assumption of linearity is made for two reasons. First, because the resulting equations are very much easier to use use and secondly, because available experimental data is not precise enough to permit identification of nonlinear effects.

We define the following linear coefficients:

<u>Frequency</u>

$$\omega^2 = k/m$$

$$\lambda^2 = \omega^2 - c^2 = \omega^2(1 - \bar{c}^2) = \omega^2 \eta^2$$

Damping

$$c = K/m$$

$$\bar{c} = c/\omega$$

$$\eta = \sqrt{1 - \bar{c}^2}$$

The peak force in the spring is $k\delta_{MAX}$. Dividing by the mass we have $\omega^2\delta_{MAX}$, which is the "Dynamic Response Index."

Note that

$$\omega^2\delta_{MAX} = \frac{k}{m}\delta_{MAX}$$

$$= \frac{\text{peak force}}{\text{effective mass.}}$$

We assume that the size of the structure (such as the spine) in which the force is reacted is proportional to the mass of the system and therefore to the weight of the subject. Thus,

$\omega^2\delta_{MAX}$ is proportional to $\frac{\text{peak force}}{\text{structural area}}$

$\therefore \omega^2\delta_{MAX}$ is proportional to peak stress.

We therefore assume that the Dynamic Response Index (DRI) is a measure of the maximum stress generated in the body. Thus, the DRI is the most logical parameter against which to correlate injury and other stress-induced physiological effects.

For zero initial conditions $\omega^2\delta_{MAX}$ is given by the following equations:

Impulsive velocity change Δv

$$\omega^2\delta_{MAX} = \omega\,\Delta v\, e^{-\frac{\bar{c}}{\eta}(\pi - \sin^{-1}\eta)} \tag{170}$$

Short period acceleration $\ddot{Y}_c$

$$\omega^2\delta_{MAX} = \ddot{Y}_c\left(1 + e^{\frac{-\bar{c}\pi}{\eta}}\right) \tag{171}$$

Now we have assumed that any line in a tolerance graph is defined by $\omega^2\delta_{MAX}$ = constant. Thus, we can equate equation 170 and 171 for any particular tolerance line, with values Δv_{CRIT} and $\ddot{Y}_{CRIT}$ respectively.

i e,
$$\omega \Delta \dot{S}_{CRIT}\, e^{-\frac{\bar{\varepsilon}}{\eta}(\pi - \sin^{-1}\eta)} = \ddot{Y}_{CRIT}\left(1 + e^{-\frac{\bar{\varepsilon}\pi}{\eta}}\right)$$

so that
$$\omega = \left(\frac{\ddot{Y}_{CRIT}}{\Delta \dot{S}_{CRIT}}\right) \frac{1 + e^{-\frac{\bar{\varepsilon}\pi}{\eta}}}{e^{-\frac{\bar{\varepsilon}}{\eta}(\pi - \sin^{-1}\eta)}} \tag{172}$$

The "corner duration" Δt_1 in figure 18 can also be obtained from equation 172, since

$$\Delta \dot{S}_{CRIT} = \ddot{Y}_{CRIT}\, \Delta t_1 \tag{173}$$

at this point. Substituting for $\Delta \dot{S}_{CRIT}$ in equation 172

$$\omega \Delta t_1 \quad \Delta \tau_1 = \frac{1 + e^{-\bar{\varepsilon}\pi/\eta}}{e^{-\frac{\bar{\varepsilon}}{\eta}(\pi - \sin^{-1}\eta)}} \tag{174}$$

This equation is plotted in figure 20.

We see that for a conventional tolerance graph, one parameter, the "corner duration" Δt_1, defines the frequency of the model <u>if the damping ratio is known</u>. However, it is impossible to deduce the damping ratio from tolerance data. This must be obtained from other classes of experiment.

b. <u>The use of impedance data to define damping and nonlinearity.</u>

Probably the simplest -- and the most accurate -- experiments which can be performed on the human body are those in which sinusoidal vibration is imposed upon a live subject and measurements are made of the "Mechanical Impedance" defined as

$$\text{Mechanical Impedance } |Z| = \frac{\text{Force input to system}}{\text{Velocity of input point}}$$

A second common experimental method involves determining Amplitude Transmissability by measuring the amplitude of oscillation of various parts of the test subject's body and dividing these readings by the amplitude of the vibration input. This is much less accurate than impedance measurements because we have to assume that transducers mounted on a subject give an accurate transcription of the motion of that part of the body to which they are attached. In practice this is usually impossible for frequencies in excess of 50 c.p.s., as shown by the work of von Gierke (Reference 4, p. 155). Significant errors may occur at much lower frequencies if the transducer attachments are not carefully designed. Nevertheless, amplitude ratio measurements are qualitatively useful because they yield a much clearer physical picture of the body's modes of oscillation and thus aid in a clearer understanding of the mechanism involved.

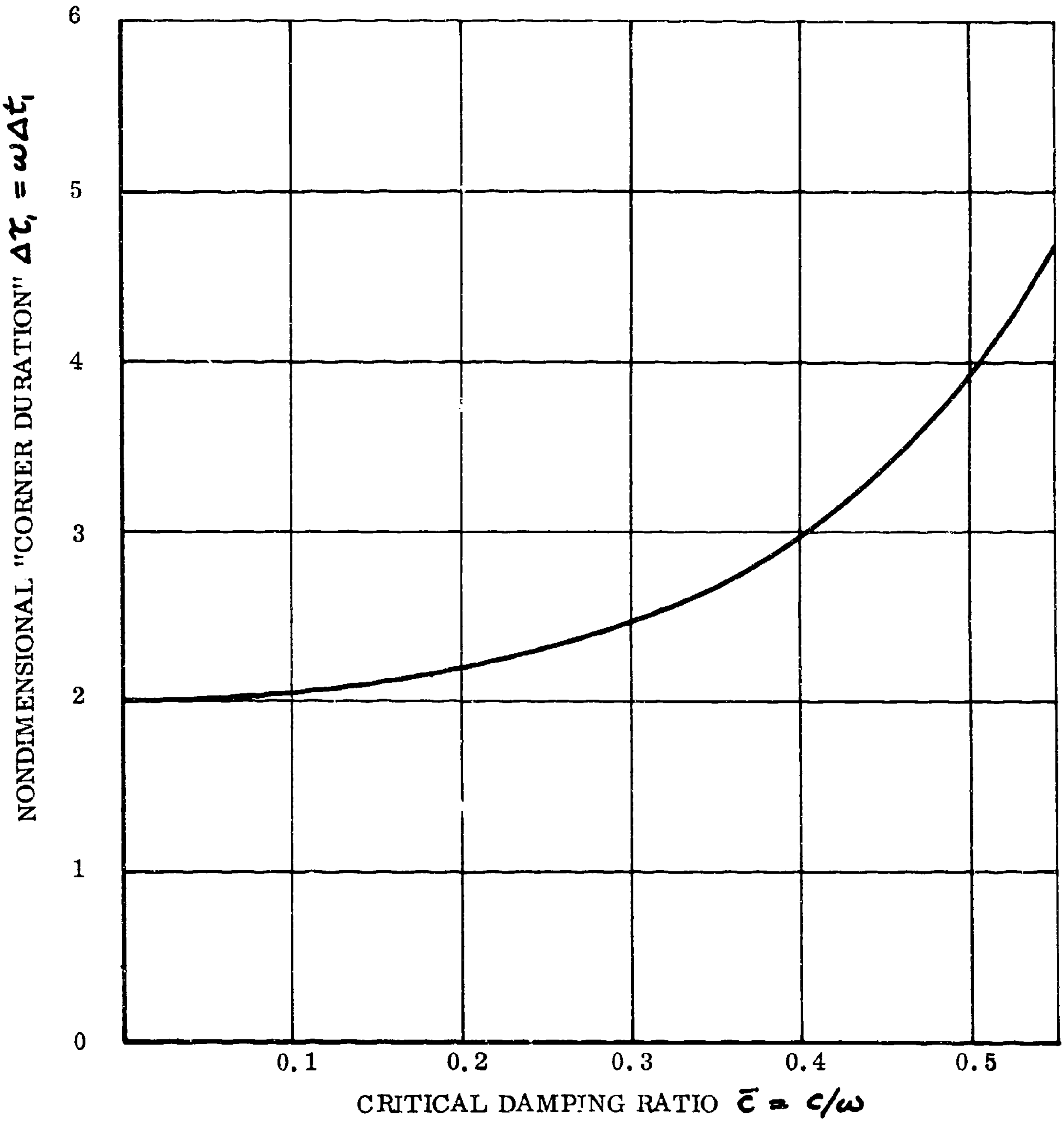


Figure 20. Variation of nondimensional corner duration $\Delta\tau_1$ with damping ratio for a rectangular acceleration pulse.

As variants of amplitude transmissability, some experimenters have used acceleration ratio. Velocity ratio is also a possibility. Both of these ratios are proportional to amplitude ratio, so long as the motion is sinusoidal. In nonlinear systems however, there may be marked differences between them, so that it is important to define the actual ratio being employed.

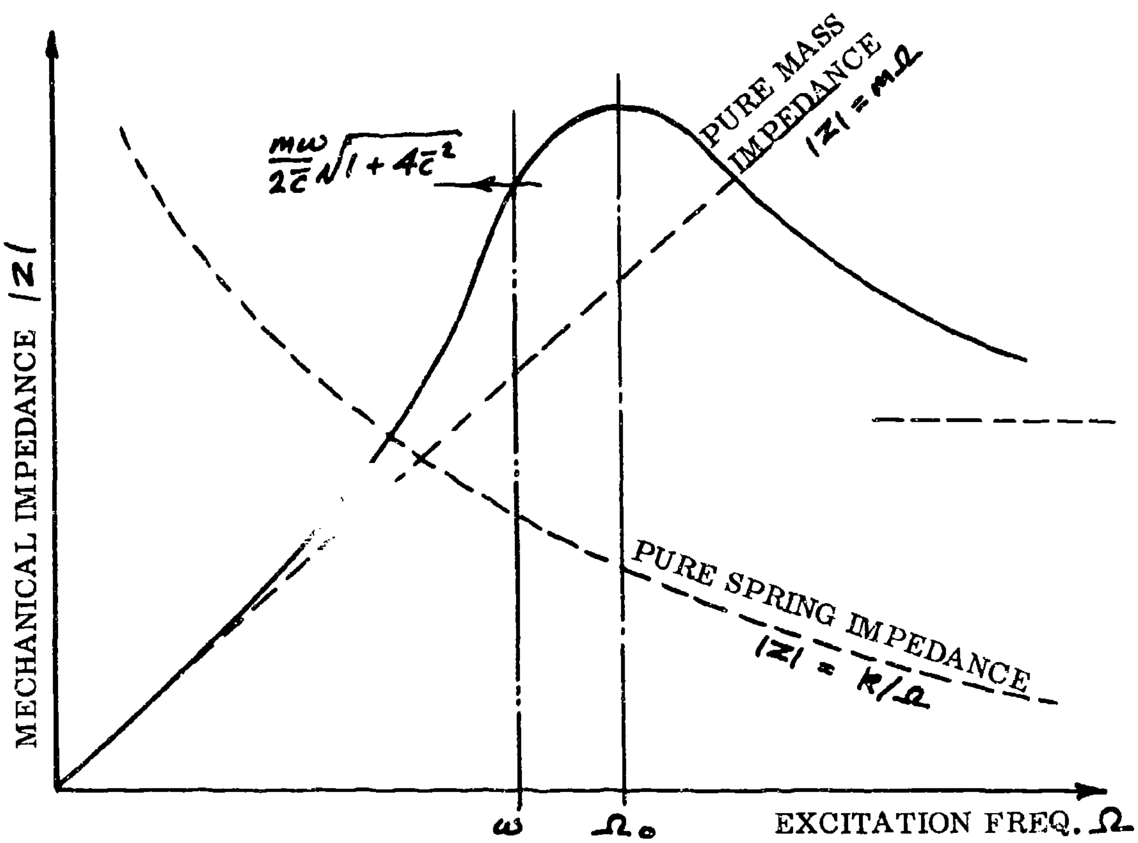


Figure 21. Mechanical impedance of a linear, single degree of freedom dynamic system excited at the spring base.

The mechanical impedance of a linear dynamic system is sketched in figure 21, and the peak value of $|Z|$ occurs at a frequency Ω_0, given by

$$\frac{\Omega_0}{\omega} = \sqrt{\frac{4\zeta^2 + \sqrt{1+8\zeta^2}}{1 - 4\zeta^2(4\zeta^2 - 2)}} \tag{175}$$

and the peak value of $|Z|$ is approximately

$$|Z|_{MAX} = \omega m \sqrt{1 + \frac{1}{4\zeta^2}} \tag{176}$$

These relationships are derived in Appendix II.

Since it is not necessary to instrument the test subject in order to obtain mechanical impedance readings, this test procedure is the most precise and at the same time the most satisfying from a scientific point of view. It should be noted, however, that resonant frequencies cannot be obtained directly from impedance data. If there is appreciable damping in the system, the frequencies at which maximum impedance occurs are considerably higher than the resonant frequency. The relationship between them is given in figure 22. The maximum impedance for a linear system is plotted in figures 23 and 24.

An impedance curve can tell us the damping in a system. From figure 21 it is obvious that we know $|Z|_{max}$ and Ω_0, the frequency at which $|Z|_{max}$ occurs. Thus, from figure 24 we can determine the damping ratio ζ, if we know the effective mass.

It should be noted that we are almost always concerned with the parameter $|Z|/m$, not $|Z|$ alone. It is strongly recommended therefore that future impedance measurements be reported in the form of $|Z|/m$, rather than $|Z|$. The former is a more fundamental parameter and should reduce scatter by eliminating the effect of weight variation between subjects. Moreover, the variability of $|Z|/m$ can be statistically analyzed, whereas a knowledge of the $|Z|$ distribution is meaningless because it may reflect nothing more than a variation in the weights of the subject tested.

Once the damping factor is known, the undamped natural frequency can be obtained from figure 22.

If amplitude transmission measurements are made, the relationship between the resonant frequency Ω_{RES} and the peak impedance frequency Ω_0 will also enable the damping ratio to be read from figure 22.

Impedance measurements with the human body reported by Coermann in Reference 5 enable preliminary estimates of the damping coefficient ζ to be made for relatively small amplitude oscillations, at frequencies up to 20 cps. The values derived by Coermann are not as accurate as they could be, however, because he has used an approximate equation for maximum impedance. In addition, he used only the peak impedance to determine ζ, whereas the ratio of the resonant frequency to the maximum impedance frequency can also be used to obtain a second reading.

For one subject (code R.C.) Coermann's data can be summarized as follows, taking the subject mass as 185/32.2 = 5.75 slugs.

	Sitting Erect (Coermann)	Sitting Erect (Latham)[6]	Sitting Relaxed	Standing Erect
Resonant frequency (cps)	5.2	5.5	4.65	5.05
Frequency for $\lvert Z \rvert_{MAX}$ (cps)	6.3		5.3	5.9
$\bar{c}$ calculated by Coermann	0.285		0.325	0.37
$\lvert Z \rvert_{MAX}$ (dyne x sec/cm x 10^6)	6.65		5.08	5.2
$\lvert Z \rvert_{MAX}$ (lb sec/ft)	465		348	356
$\lvert Z \rvert_{MAX} / \Omega_0 m$	2.005		1.82	1.672
$\bar{c}$ based on $\lvert Z \rvert_{MAX}$	0.281		0.317	0.352
Ω_0 / Ω_{RES}	1.21		1.14	1.168
$\bar{c}$ based on Ω_{RES}	0.352		0.305	0.325
Mean value for $\bar{c}$	0.306		0.316	0.349
Natural frequency $\omega / 2\pi$ (cps)	6.1		5.14	5.67

For practical purposes we may neglect the variation between sitting erect and relaxed. Thus, the final figures become

Damping factor in sitting position, $\bar{c} = 0.31$
Damping factor standing stiff-legged, $\bar{c} = 0.35$

c. The application of rebound data to determine dynamic constants.

Impedance measurements enable us to estimate the damping in the human body if we are able to excite the mode which is of importance. We cannot obtain the full deflection of the body's structure that would be experienced in an impact test, for example. It is obvious that the mean effective damping in an impact might be different from that measured in a relatively low amplitude impedance test.

Ω_0 = FREQUENCY FOR MAX. IMPEDANCE

ω_0 = UNDAMPED NATURAL FREQUENCY

$\lambda = \sqrt{\omega_0^2 - c^2} = \omega_0\sqrt{1 - \bar{\zeta}^2}$

Ω_{RES} = RESONANT FREQUENCY

$= \omega_0\sqrt{1 - 2\bar{\zeta}^2}$

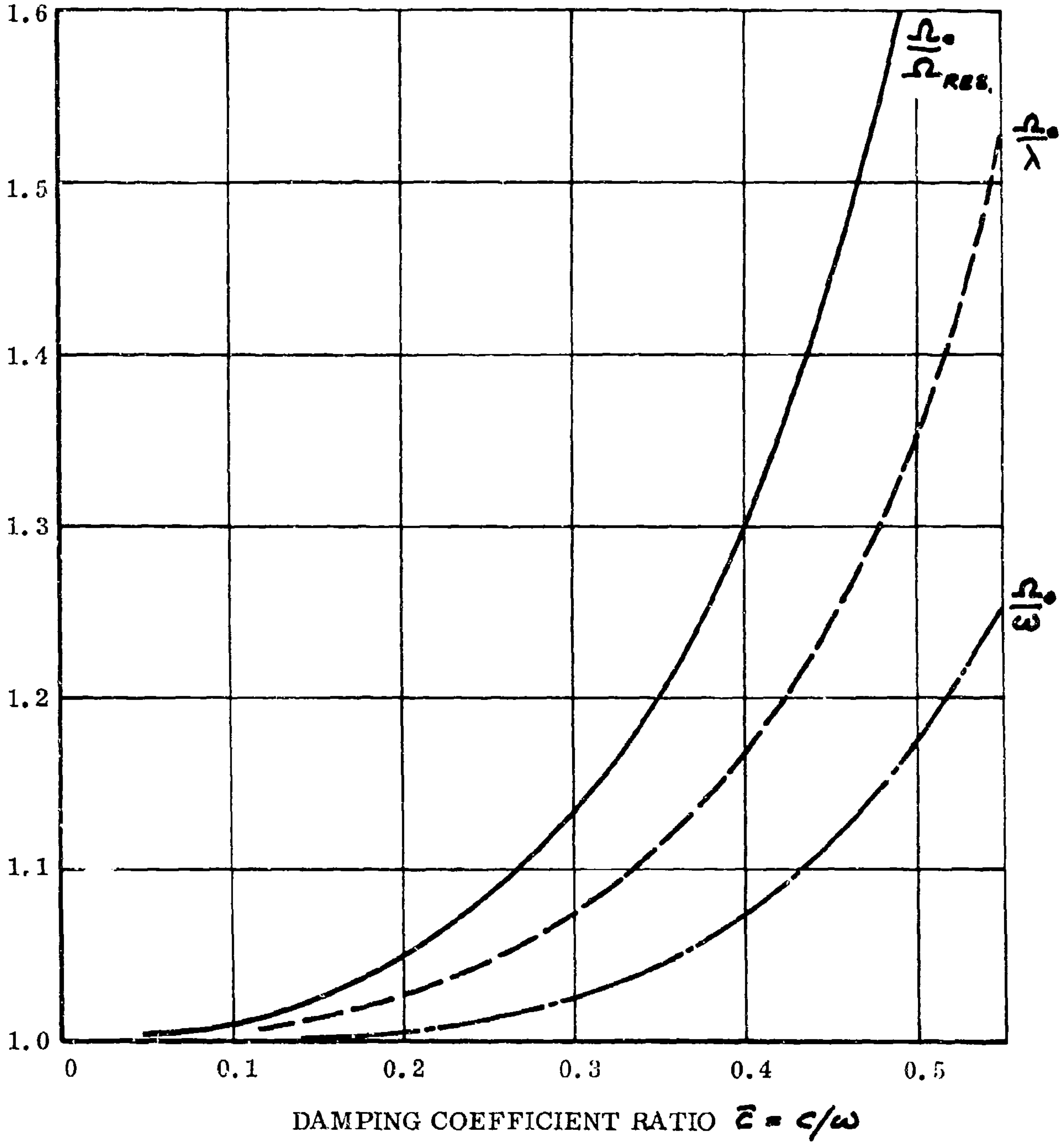


Figure 22. Variation of maximum impedance frequency ratio with damping coefficient, for a linear system.

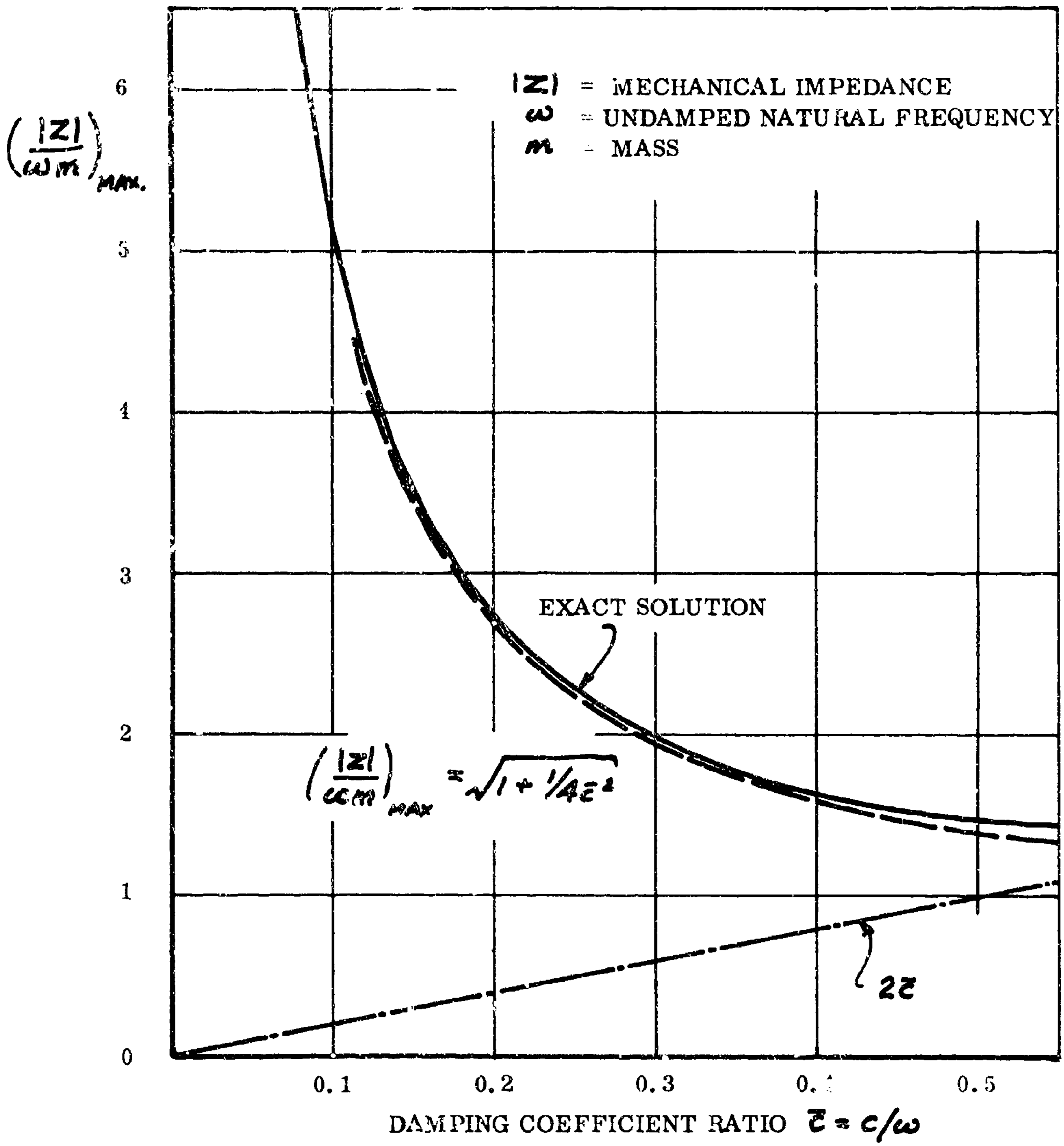


Figure 23. Variation of maximum impedance with damping ratio, in terms of the undamped natural frequency ω .

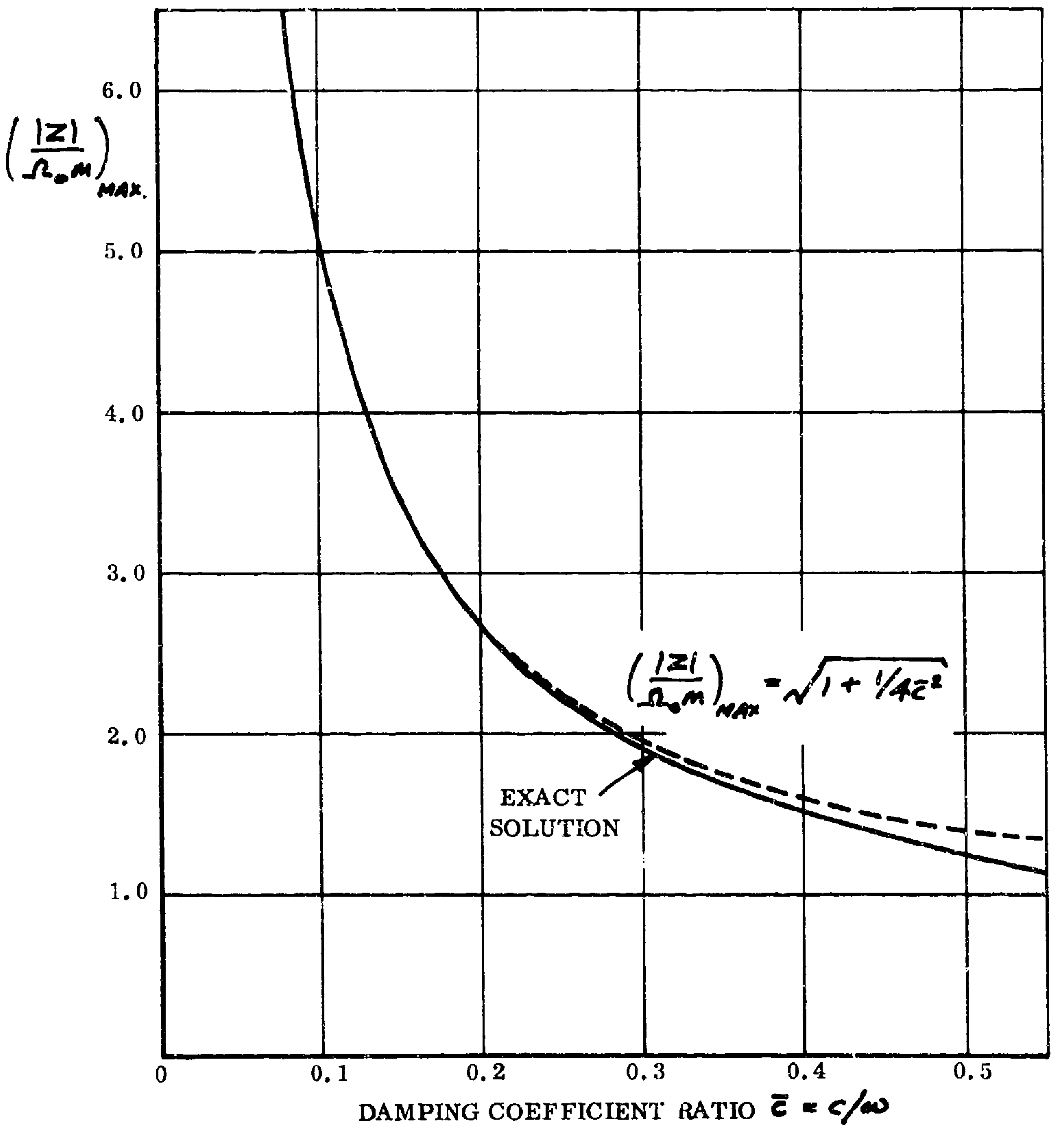


Figure 24. Variation of maximum inpedance with damping ratio, in terms of frequency Ω_0 for peak impedance.

Secondly, impedance measurements can be used to determine the degree of nonlinearity only if a suitable centrifuge is available for use in the experiments.

Rebound testing appears to offer a means of circumventing both these limitations.

(1) Rebound of a linear system.

If a linear dynamic system is subjected to an impulsive velocity change, such as by the drop test indicated in figure 25, it will rebound away from the free surface on which it impacted.

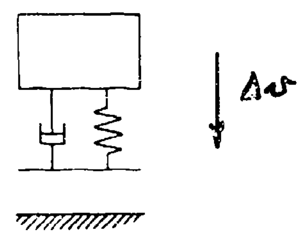


Figure 25. Impact case for a simple dynamic system.

The equation of motion is, of course,

$$\ddot{\delta} + 2c\dot{\delta} + \omega^2\delta = 0 \tag{177}$$

the initial condition being $(\dot{\delta})_0 = \Delta v$, and the solution

$$\delta = \frac{\Delta v}{\lambda} e^{-\bar{c}\omega t} \sin \lambda t \tag{178}$$

$$\dot{\delta} = \frac{\Delta v}{\eta} e^{-\bar{c}\omega t} \sin(\lambda t + \varphi) \tag{179}$$

where $\varphi = \sin^{-1}\eta \qquad \bar{c} = c/\omega \qquad \eta = \sqrt{1-\bar{c}^2} \qquad \lambda = \omega\eta$

"Take off" occurs either when the total spring and damper force falls to zero, or when the spring deflection is zero. For the first of these

$$2c\dot{\delta} + \omega^2\delta = 0 \tag{180}$$

at "take off".

Substituting equation 178 and 179 for δ and $\dot{\delta}$ equation 180 gives the result

$$n\pi - \lambda t_o = \sin^{-1} 2\bar{\varepsilon}\eta = \theta \text{(say)} \tag{181}$$

Substituting t_o into equation 179, the take off velocity is

$$\frac{(\dot{\delta})_{T.O}}{\Delta v} = \frac{e^{-\frac{\bar{\varepsilon}}{\eta}\{n\pi - \theta\}}}{\eta} \sin\{n\pi - \theta + \varphi\} \tag{182}$$

Equation 182 is plotted in figure 26, together with the velocity which exists when the spring deflection is zero.

For the case $\delta = 0$, $\sin\lambda t = 0$, so that

$$\lambda t = n\pi, \qquad t = \frac{n\pi}{\lambda}$$

$$\frac{(\dot{\delta})_{T.O}}{\Delta v} = e^{-\frac{\bar{\varepsilon}\omega n\pi}{\lambda}} \sin\{n\pi + \sin^{-1}\eta\}$$

$$= e^{-\bar{\varepsilon}\pi/\eta} \tag{183}$$

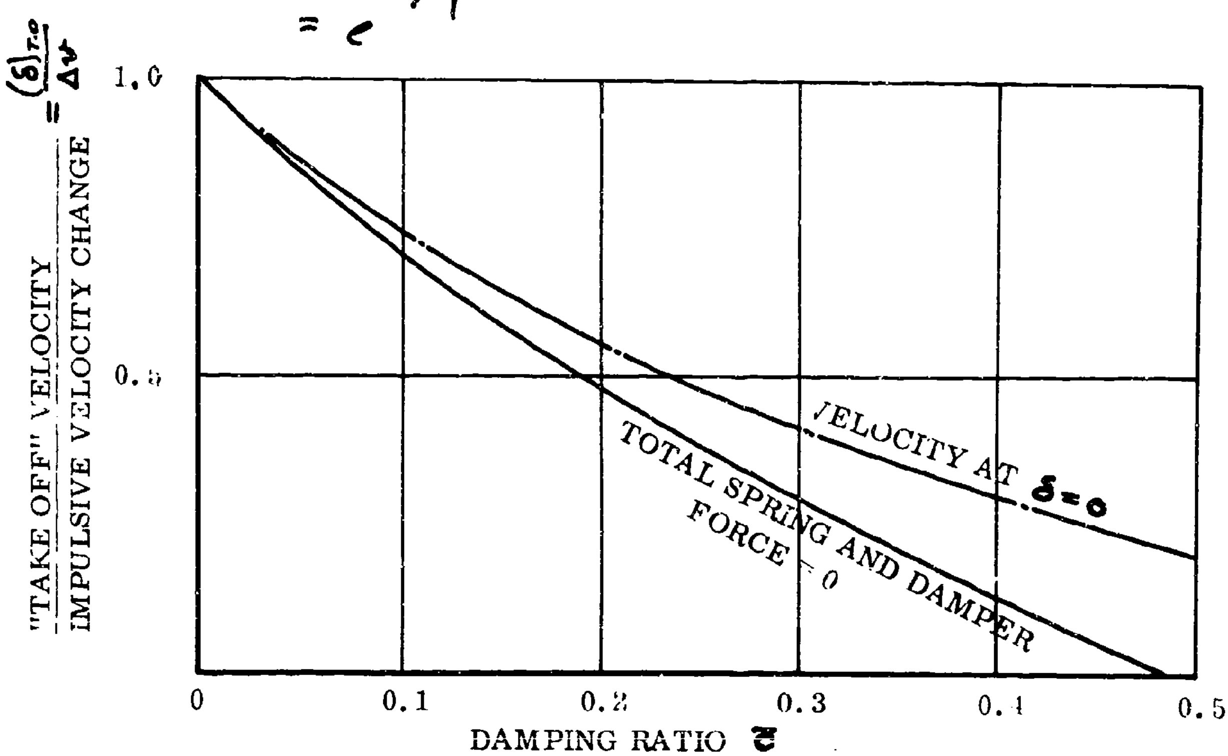


Figure 26. Variation of rebound velocity with damping in a linear system.

Consequently, if we measure the rebound velocity of live human subjects, we can immediately determine the equivalent linear damping from figure 26 or equation 183, <u>without any reference to the frequency of the system.</u>

Once we know the damping, the time to take off (t_0) gives us the frequency of the system, since from equation 181

$$\omega = \frac{\kappa\pi - \theta}{\eta t_0} \tag{184}$$

This relationship is plotted in figure 27.

Thus, we see that rebound measurements can give us a direct and precise indication of both damping and frequency, under the type of loading history with which we are most concerned in body dynamics. The theory given above is for true "bounce testing" where the subject is dropped onto a rigid surface. In Reference 12 Hirsch reports tests in which take off is caused by vertically accelerating, and then decelerating, a platform on which the subject is standing. This data is more difficult to analyze because the acceleration time history occupies a finite time. Some of Hirsch's data is plotted in figure 28, as are also boundaries for an undamped single degree of freedom system driven by rectangular pulses. It seems obvious from the relatively small scatter in Hirsch's data that good estimates of both damping and frequency could be made by using an analog to reduce this data. Also, if such tests were carried out with a carefully regularized acceleration-time profile, the damping and frequency of the subjects could be obtained by a simple analysis of the type developed earlier in this section.

(2) <u>Rebound of a nonlinear system.</u>

If the apparent frequency of the human body increases with impact velocity, during rebound testing of the type described in the previous section, we may feel reasonably confident that this is due to nonlinearities in the system, and most probably to a progressive increase of spring rate with deflection.

It should be noted that this type of experimental work offers the hope of a major breakthrough in the area of body dynamics. Consequently it is recommended that a suitable experimental program be implemented with all possible priority.

d. <u>Some examples of the analysis of existing data.</u>

At the time of writing this report, insufficient data are available to make a detailed analysis of the dynamic models which best describe the dynamic behavior of the human body. The process of collecting the available data is still far from complete. However, it is of interest to construct some provisional models in order to illustrate the use of the techniques described in the earlier sections.

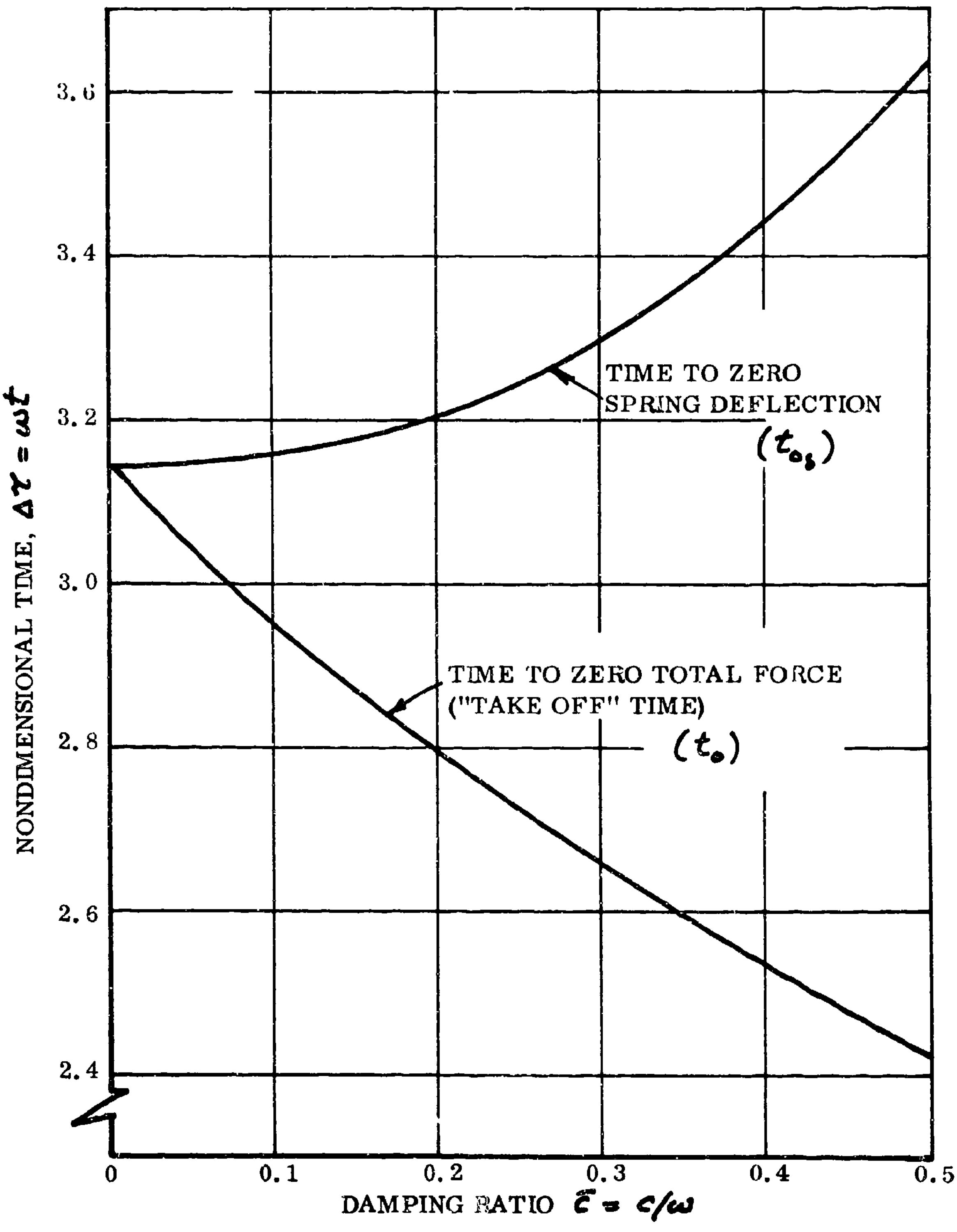


Figure 27. **Effect of model damping on time to zero force (t_o) and zero extension (t_{o_s}) for an impulsive input to a damped, linear system.**

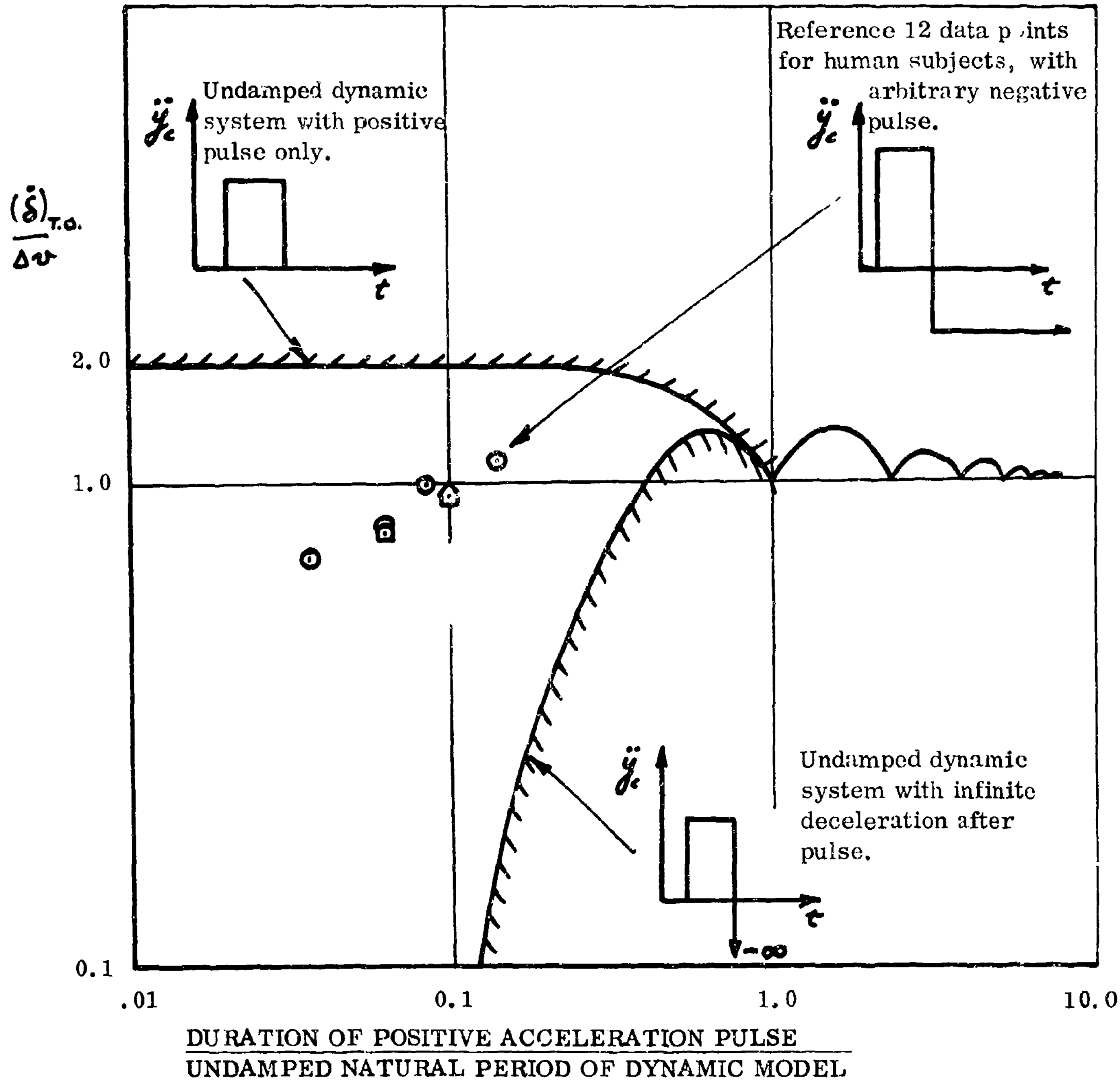


Figure 28. Take off velocity as a function of pulse duration for an undamped dynamic system, compared with Hirsch's[12] data for a standing man.

(1) Spinal model.

For the case of positive spinal acceleration, it has been found that the available data on spinal injury support the hypothesis that a single degree of freedom nonlinear damped system is a good representation of t' e human body. An estimate of the nonlinearity of the system has been made in Referei ce 10.

The mean age for U. S. Air Force aircrew personnel is 27.9 years. This age is used as a basis for estimating tolerances to acceleration.

Using data from experiments with cadaver vertebrae, and a knowledge of the variation of their strength characteristics with age (Reference 11), it can be shown that the steady-state acceleration level corresponding to 50% probability of injury for a subject aged 27.9 years is approximately 21.3g.

From Table 1 of Reference 11, corresponding to these conditions we have

$$\bar{\zeta} = 0.2245$$

$$\text{and} \quad \frac{\omega_F}{2\pi} = 8.42 \text{ cps} \tag{185}$$

Figure 20 then shows that

$$\Delta\tau_1 = 2.27$$

Since $\Delta\tau_1 = \omega_F \Delta t_1$, the real time corner duration is therefore

$$\Delta t_1 = \frac{2.27}{8.42 \times 2} = 0.0428 \text{ secs.} \tag{186}$$

The dynamic overshoot of a system subjected to a zero rise time acceleration input $\ddot{Y}_c$ and having a damping coefficient of $\zeta = 0.2245$ is, from equation 171,

$$\frac{\omega_F^2 \delta_{MAX}}{\ddot{Y}_c} = 1.485 \tag{187}$$

This means that the critical impulsive velocity change for 50% probability of injury is

$$\Delta v = \frac{\ddot{Y}_{CRIT}\,\Delta t_1}{\left(\frac{\omega_F^2 \delta_{MAX}}{\ddot{Y}_c}\right)} = \frac{21.3 \times 32.2}{1.485} \times 0.0428$$

$$= 19.75 \text{ ft/sec} \tag{188}$$

Using the values expressed in equations 186 through 188, we are able to define the 50% probability of injury curve for the case of a rectangular pulse acceleration input. This is shown in figure29.

NOTE: This injury curve is based upon dynamic models of the human body using available experimental data. It is subject to revision as additional data becomes available.

The only tolerance curves authorized by government procurement agencies for the use as design limits are those defined in HIAD or other applicable procurement specifications.

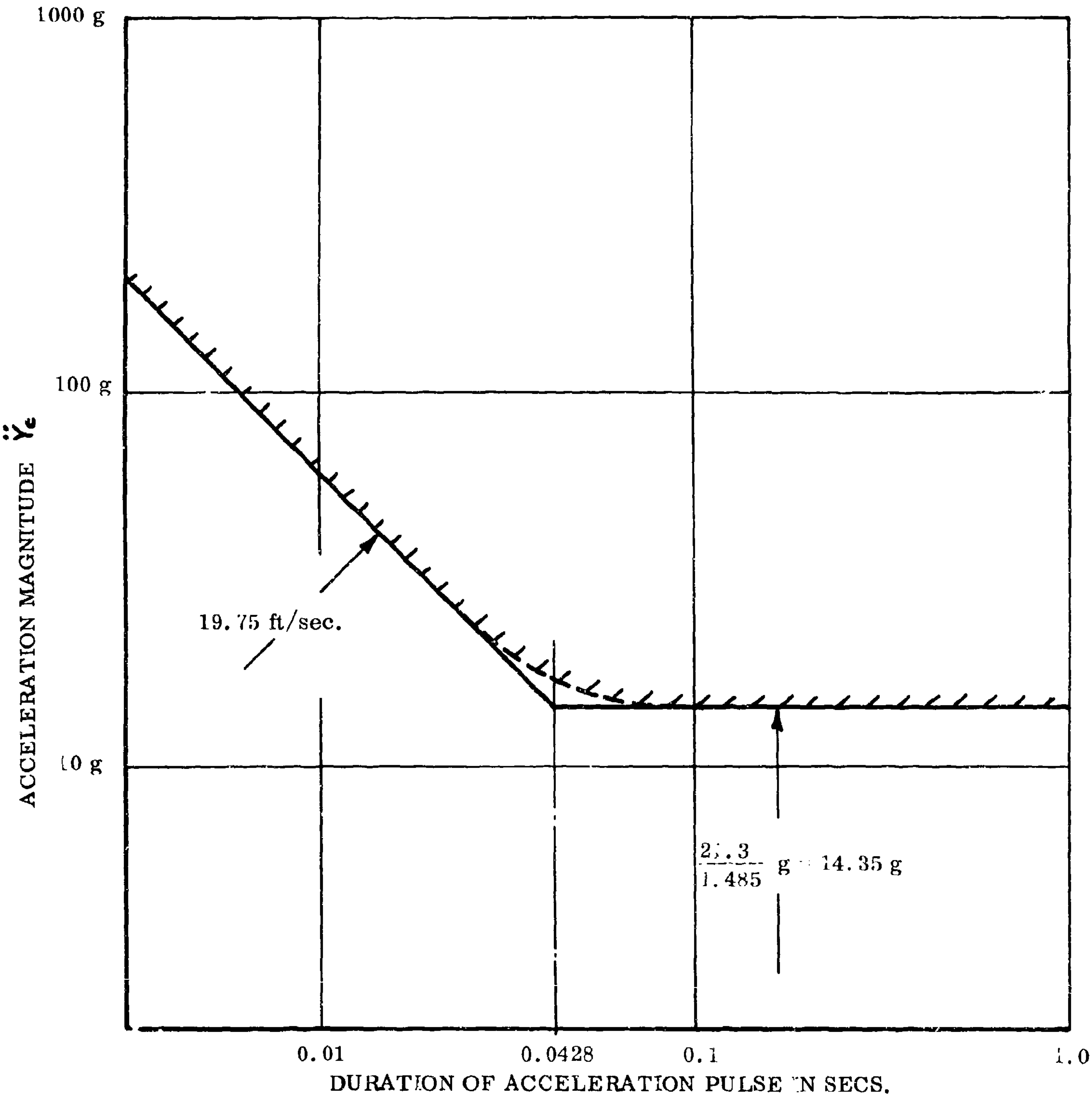


Figure 29. 50% probability of injury curve for a rectangula acceleration pulse in the spinal direction.

Summarizing the model, we have

$$\omega_F = 52.9 \text{ rad/sec}$$

$$\bar{\zeta} = 0.2245$$

Corner duration $\Delta t_1 = 0.428$ sec

Critical velocity change $\Delta v_{CRIT} = 19.75$ ft/sec $(\Delta t < \Delta t_1)$

Critical zero rise time acceleration $\ddot{Y}_{CRIT} = 14.35$ g $(\Delta t > \Delta t_1)$

It should be noted, as pointed out in Reference 11, that head involvement occurs when the duration of the acceleration pulse becomes very short. At the present time, insufficient data are available to determine the limitations associated with this degree of freedom, however.

(2) Transverse model (soft head restraint).

There is a substantial amount of data to indicate that, for the positive transverse case the 50% injury level is given by

$$\Delta v_{CRIT} = 53.6 \text{ ft/sec}$$

$$\ddot{Y}_{CRIT} = 40.0 \text{ g}$$

for rectangular acceleration pulses with zero rise time, and that a single degree of freedom dynamic model represents an adequate description. These figures will probably change somewhat as the process of data analysis continues, however.

From equation 173

$$\Delta t_1 = \frac{\Delta v_{CRIT}}{\ddot{Y}_{CRIT}} = \frac{53.6}{40.0 \times 32.2} = .0416 \text{ secs.}$$

Thus, if we know the damping ratio $\bar{\zeta}$, we could obtain the frequency from figure 20. For the probable range of values we have

$\bar{\zeta}$ =	0	0.1	0.2	0.3	0.4
$\Delta \tau_1$ =	2.0	2.6	3.45	4.72	6.83
ω =	48.1	62.5	82.9	113.4	164.1 rads/sec

Until rebound tests have been carried out, the only source of information on damping in the transverse mode is due to a single impedance curve presented by Coermann in Reference 8. This gives a peak impedance of 432 lb sec/ft at a frequency $\Omega_0/2\pi$ = 7.6 cps. Assuming that the weight of the subject was 152 lbs.,

$$\frac{|Z|_{max}}{\Omega_0 M} = 1.92$$

and from figure 24.

$$\bar{c} = 0.298$$

Since this result is for a "semisupine" position, and is a single data point, not much weight can be attached to this result. It is extremely interesting to note that the peak impedance frequency is 47.7 rads/sec., however, since this is comparable with the values deduced from pulse tolerance data.

3. DYNAMIC CONSIDERATIONS IN THE DESIGN OF EXPERIMENTS WITH LIVE HUMAN SUBJECTS.

Dynamic models are of considerable value in the design of physiological experiments, and can save a great deal of time and money. For very short acceleration durations, where velocity change only is of importance, there is little point in using a sled or HYGE accelerator, since exactly the same results can be achieved by drop testing.

The size of test equipment is also an important factor. Cases have arisen in the past where short period inputs have been attempted on equipment whose maximum working stroke automatically limits it to "impact" or impulsive velocity change experiments. Fortunately, the dynamic models so far established allow us to define test equipment parameters quite precisely. The theoretical considerations involved are examined in the next section.

a. Test equipment requirements for determining the tolerance limit of a single degree of freedom dynamic model.

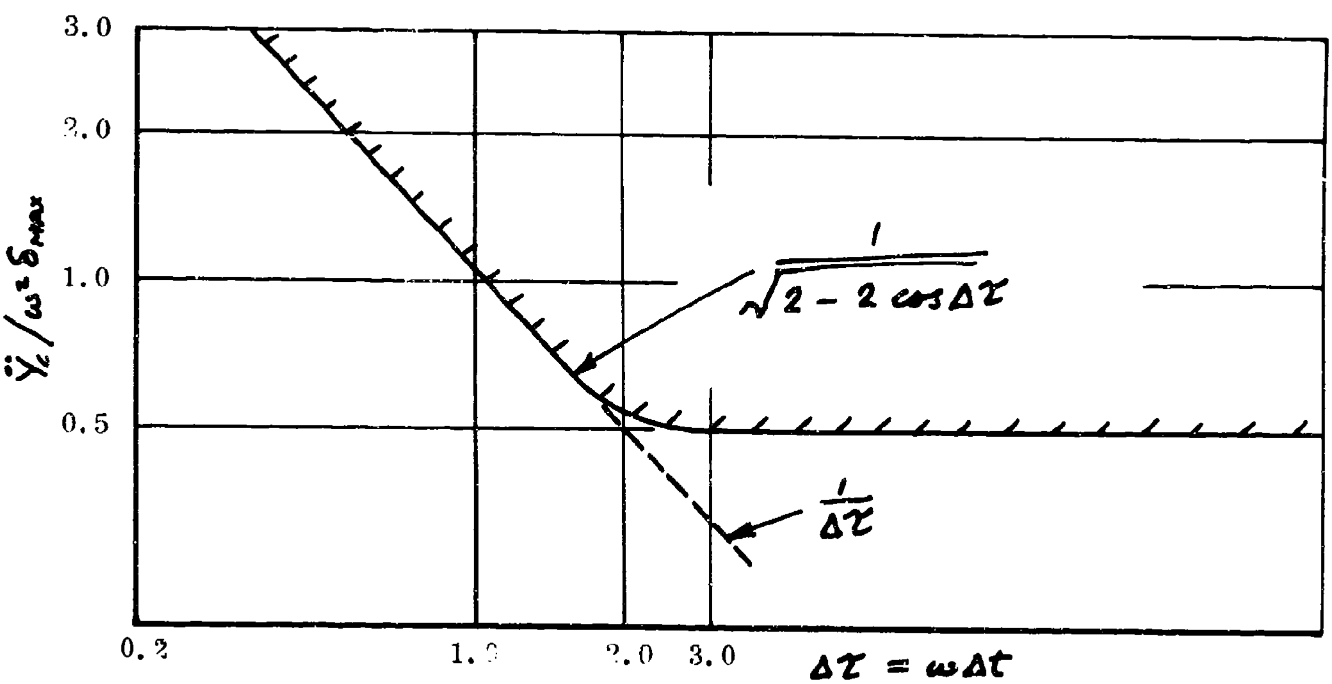


Figure 30. Rectangular pulse tolerance graph for a single degree of freedom linear dynamic system with zero damping.

Consider the nondimensional tolerance graph sketched in figure 30 for an undamped model. The velocity change associated with the rectangular pulse input upon which this graph is based will be

$$\Delta v = \ddot{Y}_c \Delta t = \omega \delta_{MAX} \left(\frac{\ddot{Y}_c}{\omega^2 \delta_{MAX}} \right) \Delta \tau \tag{189}$$

A suitable nondimensionalization for Δv is therefore

$$\frac{\omega \Delta v}{\omega^2 \delta_{MAX}} = \frac{\ddot{Y}_c \ \Delta \tau}{\omega^2 \delta_{MAX}} \tag{190}$$

Equation 190 is sketched in figure 31.

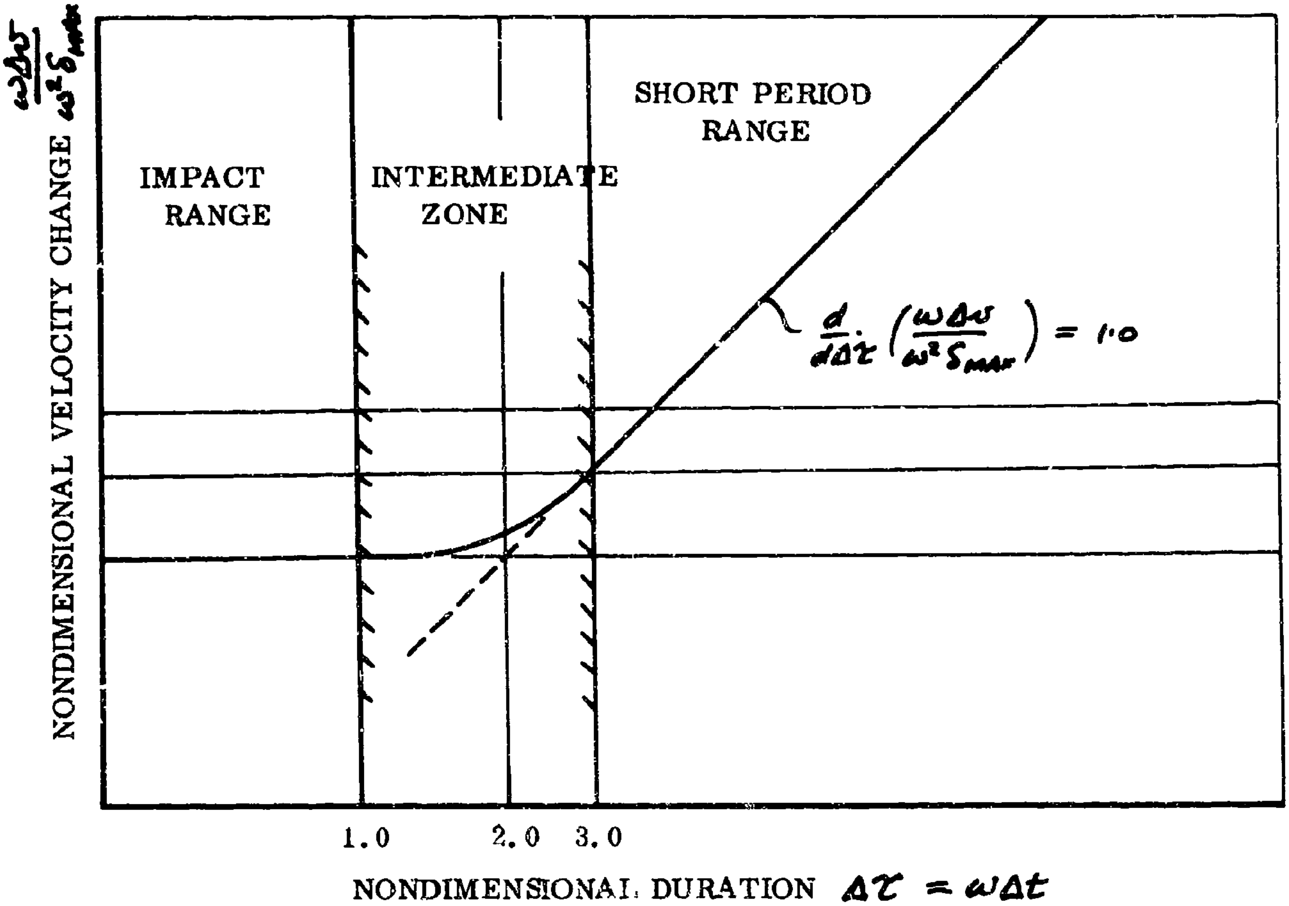


Figure 31. Nondimensional velocity change required in testing with a rectangular pulse.

It should be noted that true impace velocity changes occur for durations of $\Delta\tau < 1.0$, and true short period accelerations in the range of $\Delta\tau > 3.0$. A velocity change of at least

$$\frac{\omega \Delta v}{\omega^2 \delta_{MAX}} = 3/2$$

is required in the short period range.

The associated stroke required over the acceleration period is given by

$$S_1 = \frac{1}{2}\Delta v \Delta t = \frac{1}{2}\left(\frac{\omega \Delta v}{\omega^2 \delta_{MAX}}\right)\left(\frac{\omega^2 \delta_{MAX}}{\omega^2} \cdot \Delta\tau\right) \tag{191}$$

so that a suitable nondimensionalization is

$$\frac{\omega^2 S_1}{\omega^2 \delta_{MAX}} = \frac{1}{2}\Delta\tau\left(\frac{\omega \Delta v}{\omega^2 \delta_{MAX}}\right) \tag{192}$$

This relationship is shown in figure 32.

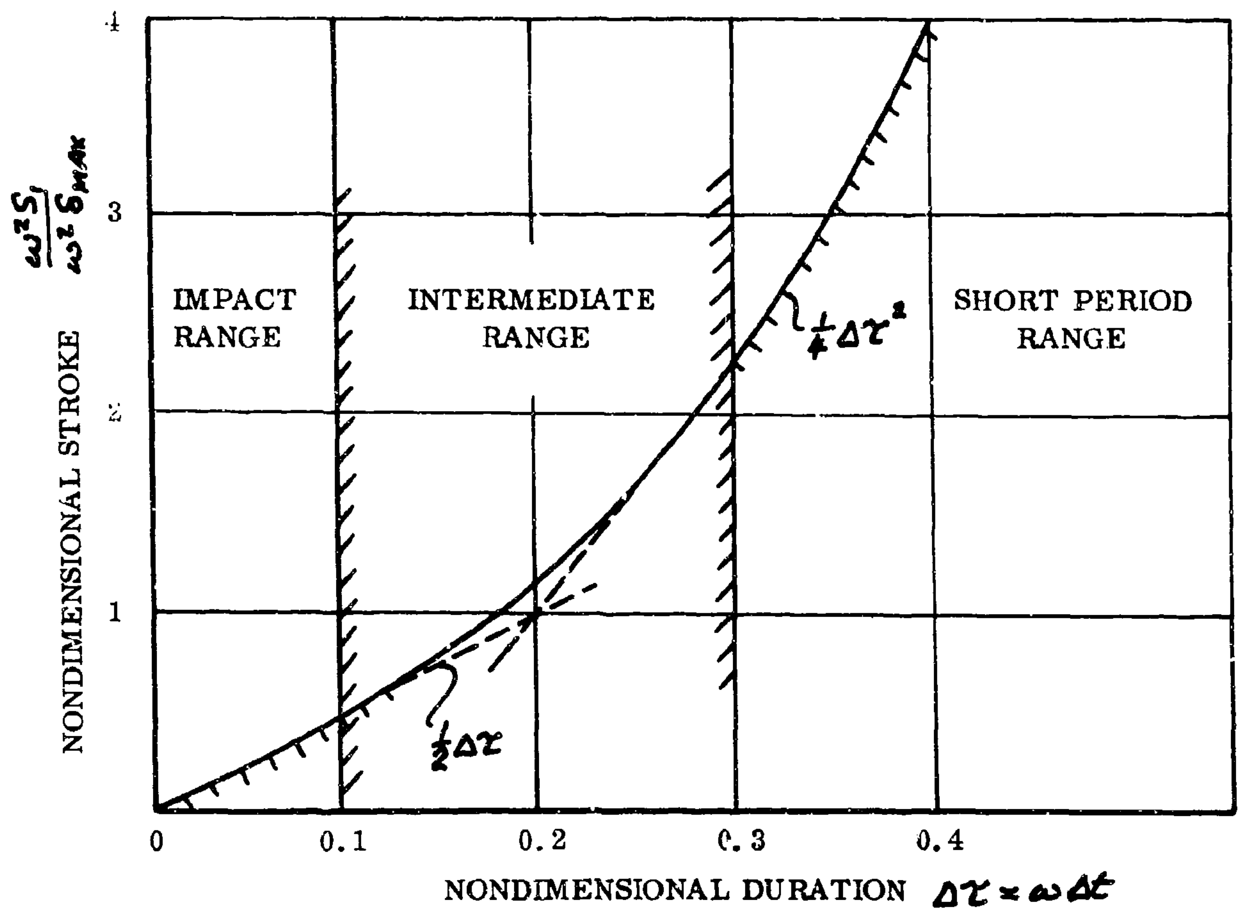


Figure 32. Variation of nondimensional accelerator stroke with acceleration duration.

Thus we can define the acceleration stroke required as follows:

$$\begin{aligned} &\underline{\text{Impact}} && \omega^2 S_1 < \frac{1}{2}\omega^2 \delta_{MAX} \\ &\underline{\text{Short-period}} && \omega^2 S_1 > \frac{9}{4}\omega^2 \delta_{MAX} \end{aligned} \tag{193}$$

We have seen that the parameter which most conveniently discriminates between impact and short period acceleration is $\Delta\tau$, the nondimensional acceleration duration.

$$\text{For impact accelerations} \quad \Delta\tau < 1$$
$$\text{For short-period accelerations} \quad \Delta\tau > 3 \tag{194}$$

The stroke of an accelerator which produces a rectangular pulse is

$$S_1 = \tfrac{1}{2}\Delta v\,\Delta t = \tfrac{1}{2}\ddot{Y}_c\,\Delta t^2 \tag{195}$$

$$= \tfrac{1}{2}\Delta v\,\frac{\Delta\tau}{\omega} = \tfrac{1}{2}\ddot{Y}_c\left(\frac{\Delta\tau}{\omega}\right)^2 \tag{196}$$

For impact experiments, $S_1 < \frac{1}{2}\frac{\Delta v}{\omega}$ (197)

For short period experiments $S_1 > \frac{9}{4}\frac{\ddot{Y}_c}{\omega^2}$ (198)

Equations 197 and 198 agree with equation 193, as we should expect. The actual test rig stroke limitations involved are plotted in figures 33 and 34 for spinal acceleration, and figures 35 and 36 for transverse, using the existing dynamic models (Reference 2). The operating range of S_1 in figures 33 through 36 is the range between the vertical cross-latched lines. Limited sample testing at levels of less than 0.1% probability of injury is not considered worthwhile. On the other hand, testing at levels of greater than 10% probability of injury is undesirable, from safety considerations. We may summarize these results as in Table 1 below:

TABLE 1

Test Rig Stroke Limitations

Impact (impulsive velocity change) Testing

Stroke must be less than	1.29 in.	for spinal (0.1% injury)
	2.24 in.	for spinal (50% injury)
	3.45 in.	for transverse (0.1% injury)
	6.70 in.	for transverse (50% injury)

Short-period Acceleration Testing

Stroke must be greater than .	3.80 in.	for spinal (0.1% injury)
	6.61 in.	for spinal (50% injury)
	8.63 in.	for transverse (0.1% injury)
	15.02 in.	for transverse (50% injury)

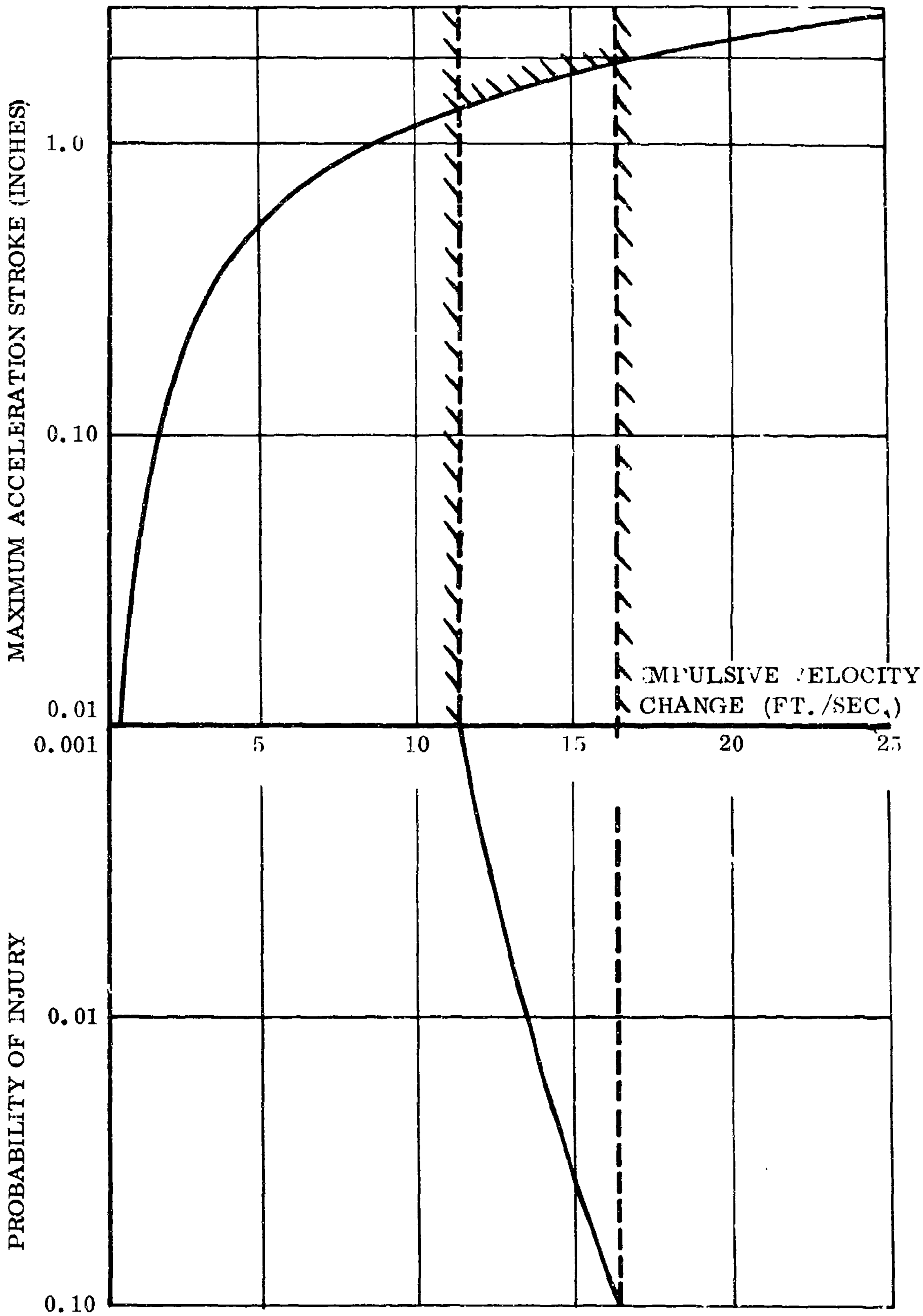


Figure 33. Impact test rig strokes for spinal acceleration of less than 0.04 sec. duration, using live human subjects.

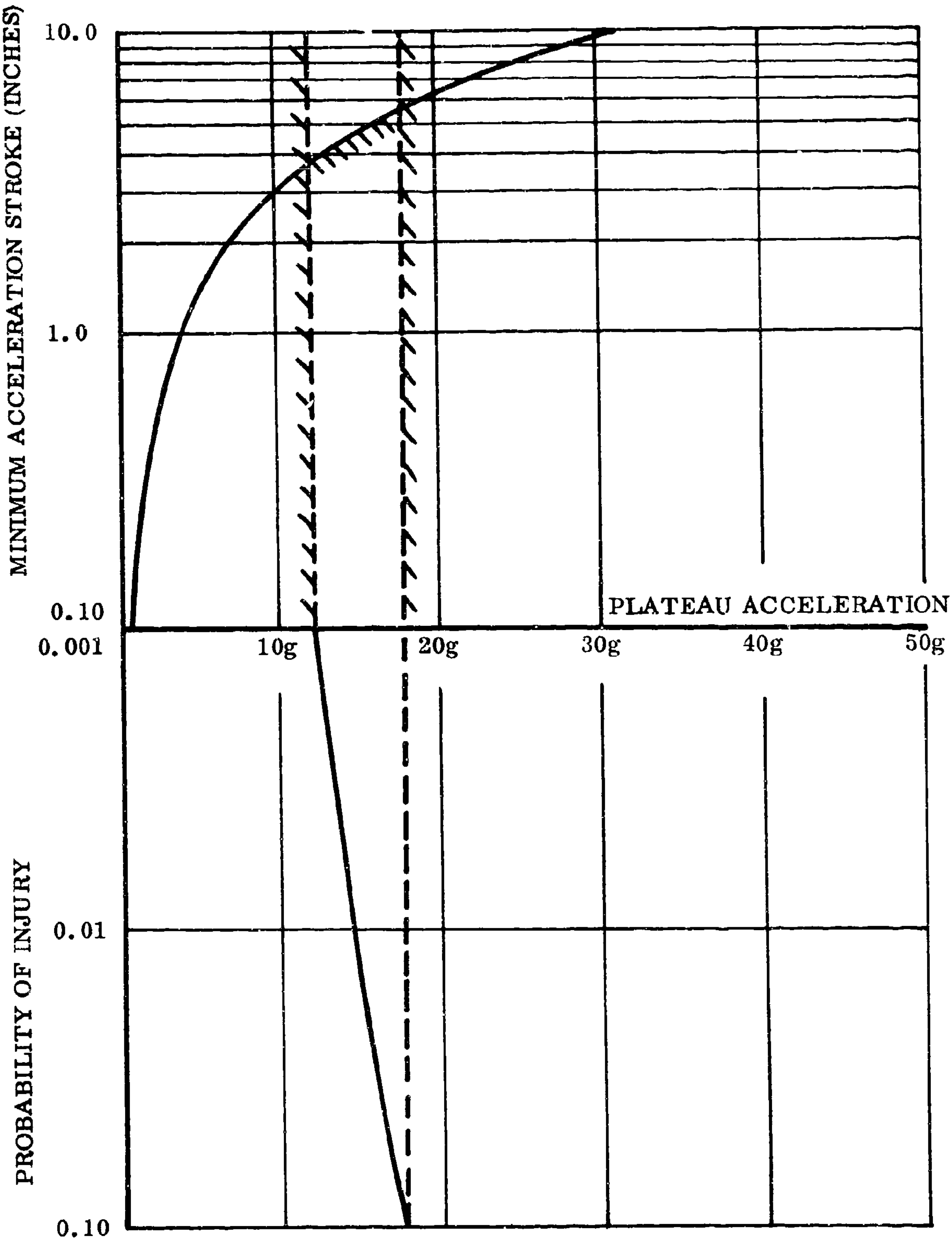


Figure 34. Short period test rig stroke for spinal acceleration of greater than 0.05 sec. duration, using live human subjects.

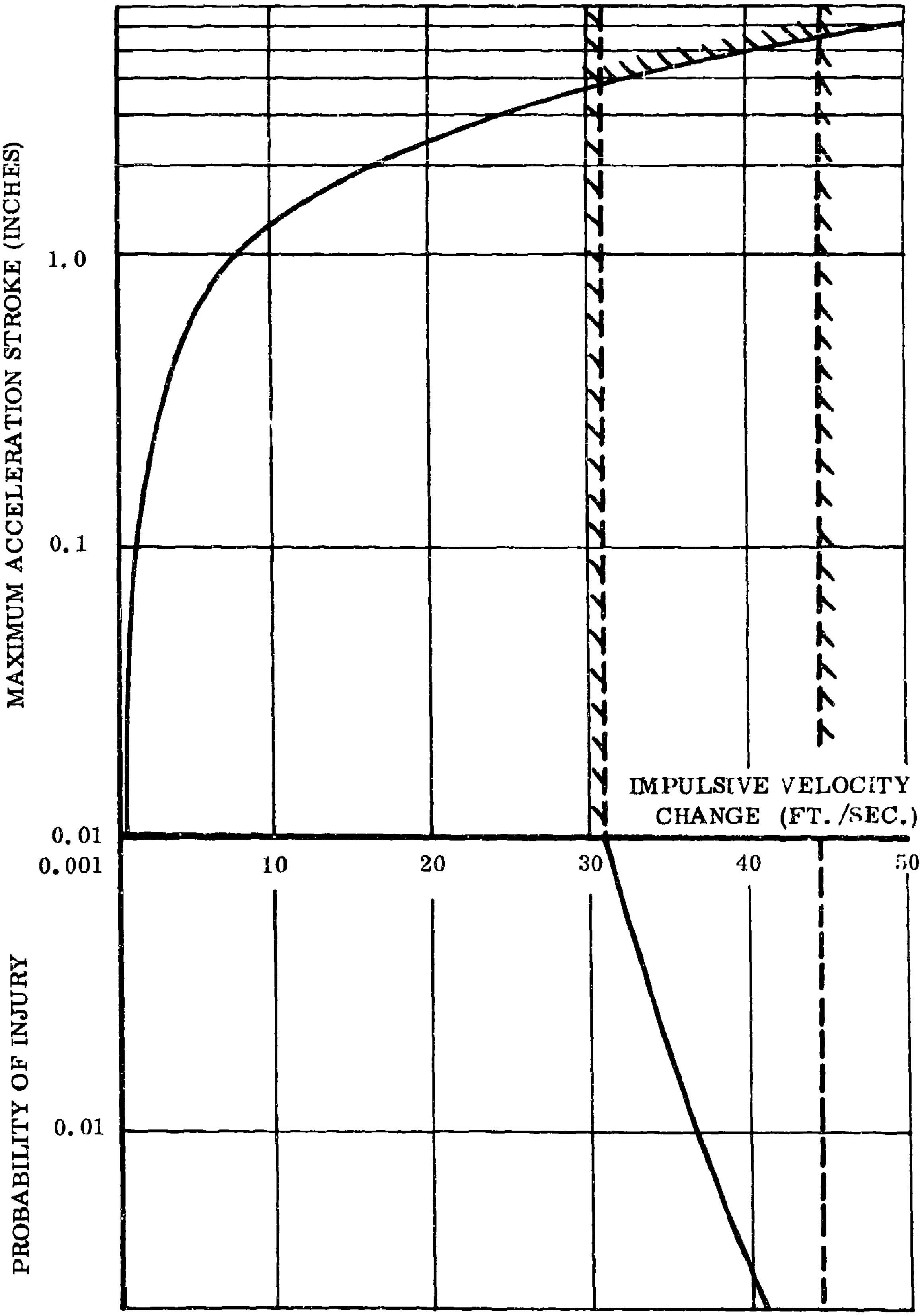


Figure 35. Short period test rig stroke for transverse testing with live human subjects.

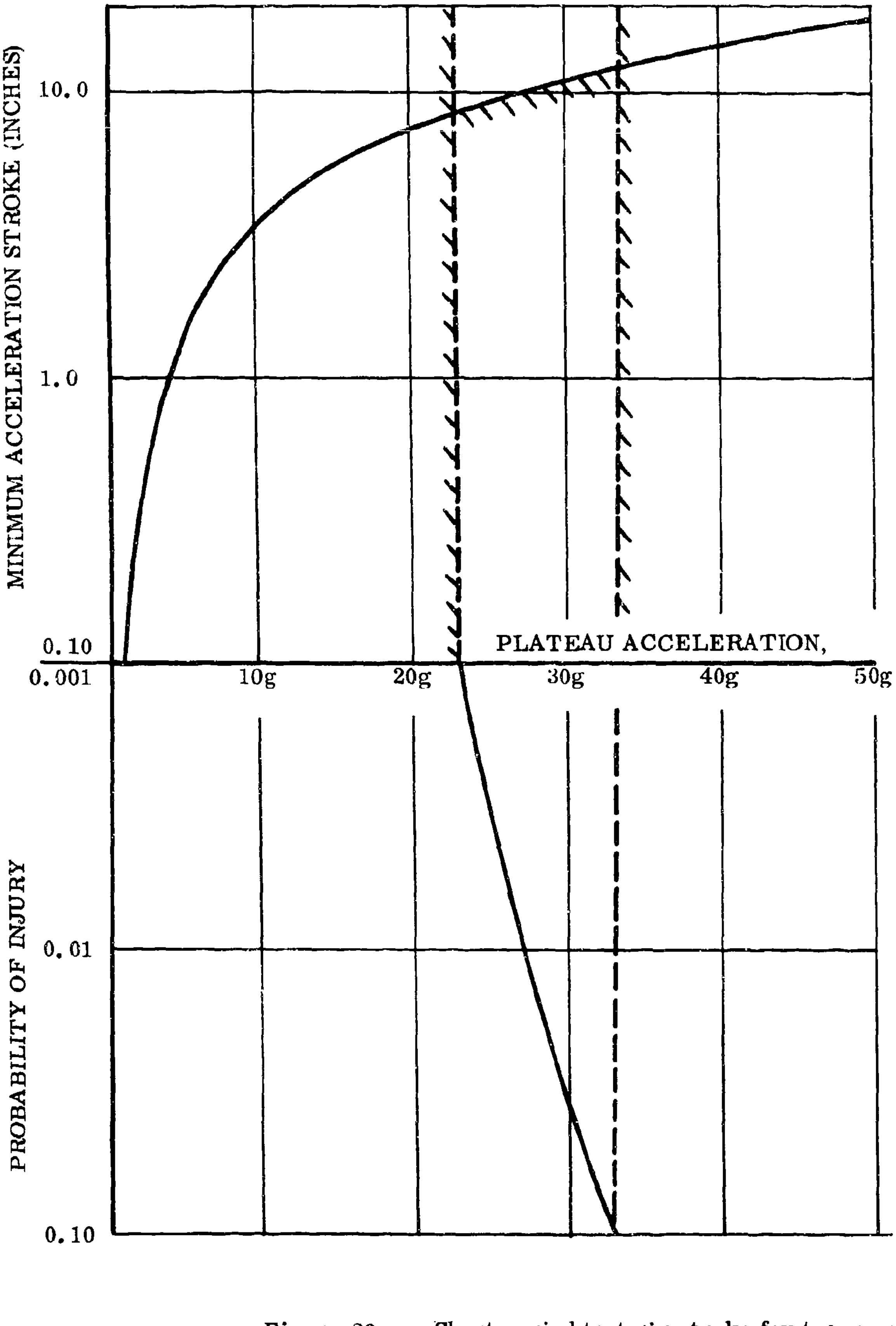


Figure 36. Short period test rig stroke for transverse testing with live human subjects.

These results are particularly significant, since <u>they indicate much more stringent test rig limitations than generally realized.</u> In the spinal case, for example, the structure supporting the test subject should be extremely rigid, and absolutely no cushioning material can be used if the results are to accurately portray the response of the human body alone. * Also of interest is the observation that a very limited stroke device -- such as a small diameter HYGE tester -- cannot be used for short-period experiments. Equally evident is the fact that "dirt drops," such as those carried out by Holcomb with the B-58 capsule, and other tests involving some resilience in either the test rig or the surface upon which it impacts, cannot be reduced to give useful data, unless the acceleration-time history is accurately recorded. Usually it is not.

In the case of transverse experiments, the reverse is true. Almost any practical support and restraint system may be used for impact testing, so long as the deflection under 40 g static load does not exceed about two inches. Correspondingly, larger strokes are required for short-period investigations and these can probably only be carried out on specialized rocket sled and similar facilities.

Thus the type of test rigs needed may be roughly summarized as follows:

	Impact	Short-Period
Spinal	Drop-Test (very stiff support)	HYGE Tester (very stiff support)
Transverse	Drop-Test	Rocket Sled, Daisy Track, Ejection Tower, Etc.

For practical purposes, the requirements in lateral testing will approximate those for transverse.

b. <u>The influence of the restraint system.</u>

A restraint system is some form of resiliency between a human body and the structure of the vehicle or vehicle subassembly with which it is associated. When the vehicle is subjected to an acceleration, the existence of the restraint system means that the acceleration-time history felt by the man is not the same as for the vehicle. This modification of the vehicle's acceleration can be favorable or unfavorable, depending upon the dynamic characteristics of the restraint system.

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* It is reported that Swearingen [9] used a hydraulic snubber in many of his "impact" experiments, so that they cannot necessarily be regarded as "impact" experiments.

A good restraint system will reduce the physiological effect of an acceleration because

(a) It increases the effective stroke through which the man is accelerated to a given velocity change, and

(b) It increases the rise time of the acceleration, thus reducing the dynamic overshoot of the man.

A bad restraint system will increase the physiological effect of an acceleration because

(a) It permits the man and the vehicle to develop relative motion before it bottoms out, thus increasing the shock effect of a given acceleration input.

The detailed effects of restraint systems, and the way in which they modify the physiological effect of input accelerations have been discussed in Part I of this report and can, in principle, at least, be considered in the analysis of test data. However, their presence is a complicating factor. At the present time, it seems logical to say that absolutely no series resiliency should be permitted in spinal tests, and that the maximum restraint deflection in lateral and transverse tests should be limited as much as possible, preferably to less than one inch, and certainly to less than two inches.

When the use of a restraint resiliency is unavoidable, its dynamic characteristics should be recorded in as much detail as possible, so that its effects can be calculated and the basic input pulse shape to the human body determined later by a device such as the Frost Restraint Analyzer.

APPENDIX I

SOLUTIONS OF THE SINGLE DEGREE OF FREEDOM LINEAR DIFFERENTIAL EQUATION WITH CONSTANT ACCELERATION INPUT

The basic equation in linear dynamics is

$$\ddot{\delta} + 2c\dot{\delta} + \omega^2\delta = \ddot{y}_c = f(t) \qquad \text{I(1)}$$

where $\ddot{y}_c$ is some function of time.

1. <u>The Complementary Function</u>

When the damping is subcritical, that is to say, when $c < \omega$ the equation

$$\ddot{\delta} + 2c\dot{\delta} + \omega^2\delta = 0 \qquad \text{I(2)}$$

has the solution

$$\omega^2\delta = e^{-ct}\left\{\omega^2(\delta)_0\cos\lambda t + \frac{\omega}{\lambda}\left[\omega c(\delta)_0 + \omega(\dot{\delta})_0\right]\sin\lambda t\right\} \qquad \text{I(3)}$$

or, if $\bar{c} = c/\omega$

$$\omega^2\delta = e^{-\bar{c}\omega t}\left\{\omega^2(\delta)_0\cos\lambda t + \frac{\omega}{\lambda}\left[\bar{c}\omega^2(\delta)_0 + \omega(\dot{\delta})_0\right]\sin\lambda t\right\} \qquad \text{I(4)}$$

where $(\delta)_0$ and $(\dot{\delta})_0$ are initial conditions and

$$\lambda^2 = \omega^2 - c^2$$

This solution is the Complementary Function in the solution of Equation I(1).

2. <u>The Particular Integral</u>

The solution to Equation I(1) contains a Particular Integral determined by the function $f(t)$.

This is given by $\delta_{P.I.} = (D^2 + 2cD + \omega^2)^{-1} f(t)$ I(5)

where D denotes the operator $\frac{d}{dt}$.

3. Constant Acceleration Input $\ddot{Y}_c$ With Sub-Critical Damping.

In the special case when $f(t) = \ddot{Y}_c$ = a constant,

$$\delta_{P.I.} = \frac{\ddot{Y}_c}{\omega^2}$$

because $D(\ddot{Y}_c)$ and $D^2(\ddot{Y}_c) = 0$. $\delta_{P.I.}$ must be added to the complementary function δ of Equation I(4) noting that the initial displacement in Equation I(4) is now $\left[\frac{\ddot{Y}_c}{\omega^2} - (\delta)_0\right]$ instead of $(\delta)_0$. Equation I(4) now becomes

$$\frac{\omega^2\delta}{\ddot{Y}_c} = 1 - e^{-\bar{c}\omega t}\left\{\left[1 - \frac{\omega^2(\delta)_0}{\ddot{Y}_c}\right]\cos\lambda t + \frac{\omega}{\lambda}\left[\bar{c}\left(1 - \frac{\omega^2(\delta)_0}{\ddot{Y}_c}\right) - \frac{\omega(\dot{\delta})_0}{\ddot{Y}_c}\right]\sin\lambda t\right\} \qquad \text{I(6)}$$

$$= 1 - \frac{e^{-\bar{c}\omega t}}{\eta_1}\left[1 - \frac{\omega^2(\delta)_0}{\ddot{Y}_c}\right]\sqrt{1 - 2\bar{c}\varphi_1 + \varphi_1^2}\ \sin(\lambda t + \theta) \qquad \text{I(7)}$$

where

$$\sin\theta = \eta_1 / \sqrt{1 - 2\bar{c}\varphi_1 + \varphi_1^2}$$

$$\varphi_1 = \frac{\omega(\dot{\delta})_0}{\ddot{Y}_c} \Big/ \left[1 - \frac{\omega^2(\dot{\delta})_0}{\ddot{Y}_c}\right]$$

$$\bar{c} = c/\omega \qquad \eta_1 = \sqrt{1 - \bar{c}^2} = \frac{\lambda}{\omega}$$

If we write

$$A = 1 - \frac{\omega^2(\delta)_0}{\ddot{Y}_c}$$

$$B = \frac{\omega}{\lambda}\left[\bar{c}\left(1 - \frac{\omega^2(\delta)_0}{\ddot{Y}_c}\right) - \frac{\omega(\dot{\delta})_0}{\ddot{Y}_c}\right] \qquad \text{I(8)}$$

The displacement δ has maximum and minimum values when the displacement velocity $\dot{\delta} = 0$. These are given when

$$e^{-\bar{c}\omega t}\left\{(\bar{c}\omega A - B\lambda)\cos\lambda t + (A\lambda + \bar{c}\omega B)\sin\lambda t\right\} = 0$$

That is, when

$$\omega\sqrt{A^2+B^2}\ \sin(\lambda t+\chi)=0 \qquad \text{I(9)}$$

where $$\sin\chi = \frac{\bar{\varepsilon}A - B(\lambda/\omega)}{\sqrt{A^2+B^2}}$$

Equation I(9) is satisfied when

$$\lambda t + \chi_1 = n\pi \qquad \text{where } n = 0,\ 1,\ 2,\ 3 \ldots.$$

i e, $$\lambda t_{MAX} = n\pi - \chi_1 \qquad \text{I(10)}$$

Now from Equation I(8),

$$\bar{\varepsilon}A - B\frac{\lambda}{\omega} = \frac{\omega(\dot{\delta})_0}{\ddot{Y}_c} \qquad \text{I(11)}$$

And $$A^2+B^2 = \frac{1}{\varphi^2}\left[\frac{\omega(\dot{\delta})_0}{\ddot{Y}_c}\right]^2 + \frac{\omega^2}{\lambda^2}\left[\frac{\omega(\dot{\delta})_0}{\ddot{Y}_c}\right]^2\left[\frac{\bar{\varepsilon}^2}{\varphi^2} - \frac{2\bar{\varepsilon}}{\varphi} + 1\right]$$

$$\sqrt{A^2+B^2} = \frac{1}{\varphi\eta_1}\left[\frac{\omega(\dot{\delta})_0}{\ddot{Y}_c}\right]\sqrt{1-2\bar{\varepsilon}\varphi+\varphi^2}$$

$$\therefore\ \sin\chi_1 = \varphi\eta_1\Big/\sqrt{1-2\bar{\varepsilon}\varphi+\varphi^2} \qquad \text{I(12)}$$

Now $$\sin\{\lambda t_{MAX}+\theta\} = \sin\lambda t_{MAX}\cos\theta + \cos\lambda t_{MAX}\sin\theta$$

$$\sin\lambda t_{MAX} = \sin(n\pi-\chi_1) = (-1)^n\sin\chi_1$$

$$= \frac{(-1)^n\varphi\eta_1}{\sqrt{1-2\bar{\varepsilon}\varphi+\varphi^2}}$$

$$\therefore \sin\{\lambda t_{MAX} + \theta\} = \frac{(-1)^n \varphi \eta_1 \cos\theta}{\sqrt{1-2\bar{c}\varphi+\varphi^2}} + \sqrt{1-\frac{\rho^2\eta_1^2}{1-2\bar{c}\varphi+\varphi^2}}\,\sin\theta$$

$$= \frac{\eta_1}{\sqrt{1-2\bar{c}\varphi+\varphi^2}}\left\{(-1)^n\varphi\sqrt{1-\frac{\eta_1^2}{1-2\bar{c}\varphi+\varphi^2}} + \sqrt{1-\frac{\varphi^2\eta_1^2}{1-2\bar{c}\varphi+\varphi^2}}\right\} \qquad \text{I(13)}$$

Substituting in Equation I(7),

$$\frac{\omega^2\delta_{MAX}}{\ddot{Y}_c} = 1 + e^{-\frac{\bar{c}}{\eta_1}\lambda t_{MAX}}\left\{1-\frac{\omega^2(\delta)_0}{\ddot{Y}_c}\right\}\left\{\sqrt{1-\frac{\varphi^2\eta_1^2}{1-2\bar{c}\varphi+\varphi^2}} + (-1)^n\sqrt{1-\frac{\eta_1^2}{1-2\bar{c}\varphi+\varphi^2}}\right\} \qquad \text{I(14)}$$

where $$\lambda t_{MAX} = n\pi - \sin^{-1}\frac{\varphi\eta_1}{\sqrt{1-2c\varphi+\varphi^2}} \qquad \text{I(15)}$$

Note that, for zero initial values of δ and $\dot{\delta}$ Equation I(7) becomes

$$\frac{\omega^2\delta}{\ddot{Y}_c} = 1 - \frac{e^{-\bar{c}\omega t}}{\eta_1}\sin(\lambda t + \phi) \qquad \text{I(16)}$$

where $\sin\phi = \eta_1 \qquad \cos\phi = \bar{c}$

The maximum value of $\frac{\omega^2\delta_{MAX}}{\ddot{Y}_c}$ is then

$$\frac{\omega^2\delta_{MAX}}{\ddot{Y}_c} = 1 + e^{-\frac{\pi\bar{c}}{\eta_1}} \qquad \text{I(17)}$$

and occurs when $t_{MAX} = \frac{\pi}{\lambda}$

APPENDIX II

THEORY OF MECHANICAL IMPEDANCE FOR A SINGLE DEGREE OF FREEDOM, LINEAR DYNAMIC SYSTEM

In this Appendix, we consider a damped, linear system, for which the equation of motion is

$$\ddot{\delta} + 2c\dot{\delta} + \omega^2 \delta = A \sin \Omega t \qquad \text{II(1)}$$

Writing equation II(1) in Laplacian form, and neglecting transients,

$$\bar{y}(p)\left[p^2 - 2cp + \omega^2\right] = \frac{A\Omega}{p^2+\Omega^2}$$

or

$$\bar{y}(p) = \frac{A\Omega}{\Omega^2\omega^2\left(1+\frac{p^2}{\omega^2}\right)\left(1+\frac{2\bar{c}p}{\omega}+\frac{p^2}{\omega^2}\right)} \qquad \text{II(2)}$$

The inverse of this is

$$\frac{\delta}{A\Omega} = \frac{\frac{1}{\Omega}\sin(\Omega t+\psi_1) + \frac{e^{-\bar{c}\omega t}}{\omega\sqrt{1-\bar{c}^2}}\sin\left(\omega\sqrt{1-\bar{c}^2}\,t+\psi_2\right)}{\sqrt{(\omega^2-\Omega^2)+4\bar{c}^2\omega^2\Omega^2}}$$

Neglecting transients, this reduces to

$$\frac{\delta}{A} = \frac{\sin(\Omega t + \chi_1)}{\sqrt{(\omega^2-\Omega^2)+4\bar{c}^2\omega^2\Omega^2}} \qquad \text{II(3)}$$

where

$$\chi_1 = \tan^{-1}\frac{2\bar{c}\,\Omega\omega}{\Omega^2-\omega^2} \qquad \text{II(4)}$$

Now if $A \sin \Omega t$ is the input acceleration, the input velocity is

$$\frac{A}{\Omega}\cos\Omega t \qquad \text{II(5)}$$

The force in the system is

$$2cm\dot{\delta} + m\omega^2\delta$$

From equation II(3)

$$\frac{\dot{\delta}}{A} = \frac{\Omega \cos(\Omega t + \varphi_1)}{\sqrt{(\omega^2 - \Omega^2) + 4\bar{c}^2\Omega^2\omega^2}} \qquad \text{II(6)}$$

From equations II(3) and II(5) the force in the system is given by

$$F = \frac{Am\left\{2\bar{c}\frac{\Omega}{\omega^2}\cos(\Omega t + \chi_1) + \sin(\Omega t + \chi_1)\right\}}{\sqrt{(1 - \Omega^2/\omega^2) + 4\bar{c}^2\Omega^2/\omega^2}}$$

$$= Am\sqrt{\frac{4\bar{c}^2\Omega^2/\omega^2 + 1}{(1 - \Omega^2/\omega^2) + 4\bar{c}^2\Omega^2/\omega^2}} \qquad \text{II(7)}$$

where

$$\sin\varphi_1 = \frac{2\bar{c}\Omega/\omega}{\sqrt{1 + 4\bar{c}^2\Omega^2/\omega^2}} \qquad \text{II(8)}$$

$$\cos\varphi_1 = \frac{1}{\sqrt{1 + 4\bar{c}^2\Omega^2/\omega^2}} \qquad \text{II(9)}$$

and $\therefore$

$$\tan\varphi_1 = \frac{2\bar{c}\Omega}{\omega} \qquad \text{II(10)}$$

Now from II(5), we have that

$$\dot{y}_c = \frac{A}{\Omega}\cos\Omega t \qquad \text{II(11)}$$

Dividing equation II(7) by II(11) the mechanical impedance is

$$Z = \Omega m\sqrt{\frac{4\bar{c}^2\Omega^2/\omega^2 + 1}{(1 - \Omega^2/\omega^2) + 4\bar{c}^2\Omega^2/\omega^2}} \quad \frac{\sin(\Omega t + \chi_1 + \varphi_1)}{\cos\Omega t} \qquad \text{II(12)}$$

Writing $p = \frac{\Omega}{\omega}$ and $q = 2\bar{c}$, the modulus is

$$|Z| = \Omega m \sqrt{\frac{p^2q^2+1}{p^2q^2+(1-p^2)^2}} \qquad \text{II(13)}$$

Obviously the phase angle is $\frac{\pi}{2} - \chi_1 - \varphi_1$ II(14)

Of interest is the value of $|Z|$ when $\Omega = \omega$, so that $p = 1$

$$\frac{|Z|_\omega}{m\omega} = \sqrt{\frac{1+q^2}{q^2}} \qquad \text{II(15)}$$

In general the equation for $|Z|_{MAX}$ is of more value than the equation for $|Z|_\omega$, since there is usually no way of knowing the correct value for ω .

From equation II(13),

$$\frac{|Z|}{m\omega} = p\sqrt{\frac{p^2q^2+1}{p^2q^2+(1-p^2)^2}} = \sqrt{\frac{p^4q^2+p^2}{p^4+(q^2-2)p^2+1}} \qquad \text{II(16)}$$

Differentiating with respect to p ,

$$\frac{d}{dp}\left(\frac{|Z|}{m\omega}\right) = \frac{(\alpha q^2-1)p^4+2p^2q^2+1}{\sqrt{(p^2q^2+1)(p^4+\alpha p^2+1)^3}} \qquad \text{II(17)}$$

where $\alpha = q^2-2$

Equating to zero to find the value for $|Z|_{MAX}$,

$$(\alpha q^2-1)(p^2)^2+2q^2(p^2)+1=0 \qquad \text{II(18)}$$

$$\therefore \left.p^2\right]_{|Z|_{MAX}} = \frac{4\bar{c}^2+\sqrt{1+8\bar{c}^2}}{1-4\bar{c}^2(4\bar{c}^2-2)} \qquad \text{II(19)}$$

Note that as $\bar{c} \rightarrow 0$, $p \rightarrow 1.0$

as $\bar{c} \rightarrow 0.77$, $p \rightarrow \infty$

Another limit of interest is $|Z|_{MAX}$ as $\bar{c} \rightarrow 0$.

From equation II(16), as $p \rightarrow 1.0$

$$\frac{|Z|_{MAX}}{\omega m} \rightarrow \sqrt{1 + \frac{1}{4\bar{c}^2}} \qquad \text{II(20)}$$

From equation II(16) note that as $p \rightarrow \infty$

$$\frac{|Z|}{\omega m} \rightarrow 2\bar{c}$$

A damped system oscillates at a frequency

$$\lambda = \sqrt{\omega^2 - c^2} = \omega \eta \qquad \text{II(21)}$$

Thus when a measurement of the frequency λ is available, as well as an impedance measurement, we may determine $\bar{c}$ from the ratio

$$\frac{\Omega_o}{\lambda} = \frac{\Omega}{\omega} \cdot \frac{1}{\eta} = \frac{p}{\eta} \qquad \text{II(22)}$$

In many cases the frequency for maximum amplitude of the test subject (resonance) is also determined.

Since $\delta = \delta_o - (y_1 - y_c)$

$$\left.\begin{aligned} y_1 - \delta_o &= y_c - \delta \\ \dot{y}_1 &= \dot{y}_c - \dot{\delta} \\ \ddot{y}_1 &= \ddot{y}_c - \ddot{\delta} \end{aligned}\right\} \qquad \text{II(23)}$$

The transmission factors (output/input) are therefore

Displacement: $\quad T_{\delta} = 1 - \delta / y_c$

Velocity: $\quad T_{\dot{\delta}} = 1 - \dot{\delta} / \dot{y}_c$ $\qquad$ II(24)

Acceleration: $\quad T_{\ddot{\delta}} = 1 - \ddot{\delta} / \ddot{y}_c$

From equation II(3),

$$\left.\begin{aligned} \delta &= \frac{A \sin(\Omega t + \chi_1)}{\omega^2 \sqrt{(1-p^2)^2 + \delta^2 p^2}} \\ \dot{\delta} &= \frac{p A \cos(\Omega t + \chi_1)}{\omega \sqrt{(1-p^2)^2 + \delta^2 p^2}} \\ \ddot{\delta} &= \frac{-p^2 A (\Omega t + \chi_1)}{\sqrt{(1-p^2)^2 + \delta^2 p^2}} \end{aligned}\right\} \qquad \text{II(25)}$$

Also

$$\left.\begin{aligned} \ddot{y}_c &= A \cos \Omega t \\ \dot{y}_c &= -\frac{A}{\Omega} \sin \Omega t \\ y_c &= \delta_0 - \frac{A}{\Omega^2} \cos \Omega t \end{aligned}\right. \qquad \text{II(26)}$$

Substituting equations II(25) and II(26) into II(24), the modulus of the transmission factor becomes

$$T = 1 \pm \frac{p^2}{\sqrt{(1-p^2)^2 + \delta^2 p^2}} \qquad \text{II(27)}$$

that is, the transmission factor is the same for acceleration, velocity or displacement, so long as the motion is sinusoidal.

We find the "resonant frequency" for which T is a maximum by differentiating equation II(27), equating to zero, and solving for p.

This gives us $\quad p_{RES} = \dfrac{\Omega_{RES}}{\omega} = \dfrac{1}{\sqrt{1 - 2\varepsilon^2}}$

In other words, the resonant frequency is higher than the undamped natural frequency ω, although the free oscillation frequency λ is lower.

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13 ABSTRACT

Like any other complex dynamic system the human body responds in a complex way to acceleration inputs which vary rapidly with time. The need to avoid stresses large enough to cause injury to the body usually imposes limits on the permissible input acceleration. The restraint system interposed between a vehicle and its occupant can modify the physiological effects of a vehicle's acceleration - time history. This modification should be made as favorable as possible by minimizing the stresses generated in the vehicle's occupant. To determine optimum dynamic characteristics for the restraint system, its important characteristics, and those of the human body, need to be represented in terms of a mathematical or "dynamic" model. Through suitable analysis, either mathematical or by means of a computer, those dynamic characteristics of the restraint system can be determined which will minimize the peak stresses developed in its human occupant. In this report a general theory of suitable dynamic models is developed for this type of problem. Closed form solutions for a number of simple cases are presented also. In addition a method is shown which permits development of simple dynamic models for the human body utilizing existing experimental data.

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